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MATHEMATICS OF FINANCE

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PREFACE

It is the main purpose of this book to present a teachable elementary course in the application of mathematics to a broad class of financial problems.

The material for the book has been obtained from many sources and tested in the teaching of such courses at three large universities. Experience has shown that the material is especially adapted to the needs of the students in schools and colleges of commerce and business administration, although general liberal arts and engineering students also find the course of much value.

The man with a liberal business education should surely be thoroughly and accurately trained in the operation of interest in relation to finance. This course is designed to supply such training. In particular, this book treats of the relation of interest to the amortization of debts, to the creation of sinking funds, to the treatment of depreciation, to the valuation of bonds, to the accumulation of funds in building and loan associations, and to the elements of life insurance.

Three chapters are devoted to an introduction to the elements of the mathematics of insurance. This is not a technical actuarial treatment of insurance, but simply a sufficient introduction to insurance so that the general business man who studies the book may obtain proper quantitive knowledge about the first principles of life insurance; and, as a student, may come to appreciate the beautiful system of long time finance involved in legal reserve life insurance.

For the study of the book, no mathematical preparation, except that usually included in the high school course, is absolutely necessary; but courses in freshman and even in sophomore college mathematics will be found very useful, especially if only a short time is devoted to the work on this book.

The plan of the book is such as to afford much elasticity in the time required to cover the work with a class. This is ac-

complished in part by the inclusion of many applied problems that should be solved by students when this work is given as a full course of, say three hours per week for a year. When the work is given as a briefer course, many problems may be omitted. Indeed, the entire work beginning with the chapter on probability, or with the previous chapter on building and loan associations may be omitted, without destroying the continuity of the course. These omissions would make possible a very brief course. Answers are given to some of the problems to meet the needs of those teachers who find them useful.

The necessary subjects in pure mathematics beyond the elements of algebra are given in the body of the text, in foot-notes or in special chapters at the end of the text. These final chapters are on logarithms and progressions. Students who have not studied these subjects will do well to take portions of these chapters first.

Some rather complicated formulas are developed in the book. It is important to guard the student against the tendency to substitute half blindly in the formulas. In fact, it is one of the features of our treatment to stress the formulas which the student should think out from first principles rather than the formulas most convenient for substitution without thinking.

From experience in teaching the subject, we feel justified in the view that the careful study of the course presented in this book will do much to create in the business student an appreciation of exact science in business.

The authors gratefully acknowledge their indebtedness to Professor C. H. Forsyth of Dartmouth College, to Professors J. W. Glover and H. C. Carver of the University of Michigan, to Professor H. W. Kuhn of Ohio State University, to Professor E. J. Miles of Yale University, to Professor J. F. Reilly of the University of Iowa, to Professor W. J. Rusk of Grinnell College, and to Professor E. J. Townsend of the University of Illinois for suggestions that have resulted in improvements.

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MATHEMATICS OF FINANCE

CHAPTER I

INTEREST *

1. Meaning of interest. It is a well-known fact that both individuals and business organizations will pay more than P units of money at some future date for P of those units today. The excess payment is **interest.** In a more general sense interest means a consideration for the use of capital during an interval of time.

The capital is called the **principal** and is ordinarily expressed in money, but may be expressed in any units of value.

2. The rate of interest. The **rate of interest** is the ratio of interest earned in the unit of time (usually one year) to the principal. In other words, the rate of interest is the interest earned on a unit of principal in a unit of time.

In many commercial transactions, interest is expressed as a rate per hundred instead of a rate per unit. Thus, 6 per cent means $\frac{6}{100}$ ($= 0.06$) per unit of principal. When nothing is said to the contrary, it should be understood that the rate quoted is for one year.

3. Simple interest. Simple interest on any principal is found by multiplying together the numbers which represent the principal, the rate, and the time in years.

4. Fundamental relations for simple interest.

Let P be the principal,

i the rate,

n the time.

* In the case of students who have not studied logarithms, Chapter XI should precede this chapter.

Then, by definition, the interest I is given by

$$I = Pni. \tag{1}$$

We can solve a variety of simple interest problems by the use of this relation, since we can solve it for any one of the four letters as an unknown in terms of the remaining letters.

We may state the additional relation

$$S = P + I, \tag{2}$$

where S is the amount or sum at the end of the time. From the relations (1) and (2), we can find any two of the unknowns in terms of the remaining letters. With these two fundamental relations, the important problems of simple interest can be solved.

EXERCISES

1. Given $I = Pni$; solve in turn for P, n, and i in terms of the remaining letters.

2. Given $S = P + I$, and $I = Pni$; express P in terms of S, n, and i.

3. Find the principal that will amount to \$1,400 in 2 years at 6 per cent simple interest. *Ans.* \$1,250.

4. Find the time required for \$2,000 to yield \$250 in simple interest at 5 per cent. *Ans.* 2½ years.

5. Find the rate of simple interest for 2 years and 6 months required to make \$1,500 amount to \$1,800. *Ans.* 8 per cent.

5. Computation of simple interest for part of a year. As we shall see later, simple interest is suitable only for short periods of time. In practical applications many simple interest calculations are made for intervals of time measured in days or for fractional parts of a year. It is the practice in many business transactions to calculate the interest for any number of days less than 360 on the basis of 360 days in a year. When the calculations for part of a year are based on 360 days in a year, the result is called **ordinary simple interest.** When the calculations are based on 365 days in a year the result is called **exact simple interest.**

Let d be the number of days for which interest is to be found. With the notation given in Art. 4, except that I shall be the ordinary interest and I' the corresponding exact interest, we have

$$I = \frac{Pid}{360}, \tag{3}$$

$$I' \doteq \frac{Pid}{365}, \tag{4}$$

Dividing the members of (3) by corresponding members of (4), we have

$$\frac{I}{I'} = \frac{365}{360} = \frac{73}{72},$$

that is,
$$I = \frac{73}{72} I'. \tag{5}$$

Thus the ordinary interest for any given number of days is $\frac{73}{72}$ times the exact interest.

In the calculation of interest by the use of 360 days to the year, there is much variation in practice with respect to finding the number of days from one date to another. In some financial institutions, including some large banks and trust companies, it is the practice to find the exact number of days. Thus, they would use 199 days for the time from April 1 to October 17. In other financial institutions of the same character, and in many transactions among individuals, the time would be found first in months and days, and would then be reduced to days using 30 days to the month. The practice of using 30 days to the month is clearly consistent with the use of 360 days to the year. Under this plan, the time from April 1, to October 17 is 6 months, 16 days, which gives 196 days in place of the exact 199 days. The practice of counting 30 days to a month results in making ordinary interest slightly smaller in some cases and larger in other cases, than exact interest.

6. Tables for calculation of simple interest. Although it is a very simple matter to calculate the simple interest on any principal, those who have to make many calculations usually have interest tables which show the interest on a unit of principal for various rates of interest and periods of time. The following brief table will serve to illustrate the use of a table for finding both ordinary and exact simple interest.

TABLE SHOWING THE ORDINARY AND EXACT SIMPLE INTEREST ON $1,000 AT 1 PER CENT FOR THE NUMBER OF DAYS INDICATED

Days	Ordinary Interest	Exact Interest	Days	Ordinary Interest	Exact Interest
1	0.0277778	0.0273973	150	4.1666667	4.1095890
2	0.0555556	0.0547945	160	4.4444444	4.3835616
3	0.0833333	0.0821918	170	4.7222222	4.6575342
4	0.1111111	0.1095890	180	5.0000000	4.9315068
5	0.1388889	0.1369863	190	5.2777778	5.2054795
6	0.1666667	0.1643836	200	5.5555556	5.4794521
7	0.1944444	0.1917808	210	5.8333333	5.7534247
8	0.2222222	0.2191781	220	6.1111111	6.0273973
9	0.2500000	0.2465753	230	6.3888889	6.3013699
10	0.2777778	0.2739726	240	6.6666667	6.5753425
20	0.5555556	0.5479452	250	6.9444444	6.8493151
30	0.8333333	0.8219178	260	7.2222222	7.1232877
40	1.1111111	1.0958904	270	7.5000000	7.3972603
50	1.3888889	1.3698630	280	7.7777778	7.6712329
60	1.6666667	1.6438356	290	8.0555556	7.9452055
70	1.9444444	1.9178082	300	8.3333333	8.2191781
80	2.2222222	2.1917808	310	8.6111111	8.4931507
90	2.5000000	2.4657534	320	8.8888889	8.7671233
100	2.7777778	2.7397260	330	9.1666667	9.0410959
110	3.0555556	3.0136986	340	9.4444444	9.3150685
120	3.3333333	3.2876712	350	9.7222222	9.5890411
130	3.6111111	3.5616438	360	10.0000000	9.8630137
140	3.8888889	3.8356164			

For any specified rate and for any amount, the tabular values should be multiplied by the appropriate numbers. Thus for \$5200 at 7 per cent, multiply the tabular value by 5.2 and the resulting product by 7.

EXERCISES AND PROBLEMS

1. Find the ordinary simple interest on \$2,300 at 7 per cent for 86 days.

SOLUTION: By the table, the interest on \$1,000 for 80 days at 1 per cent is \$2.2222, and for 6 days it is \$0.1667. Hence the interest at 1 per cent on \$1,000 for 86 days is \$2.3889.
Therefore, the interest on \$2,300 for 86 days at 7 per cent is

$$2.300 \times 2.3889 \times 7 = \$38.46.$$

2. Find the exact interest for the data of exercise 1. *Ans.* \$37.93.

3. Find to the nearest cent the ordinary simple interest on \$1,462.50 for 100 days at 8 per cent. Investigate by trial how few significant figures of the tabular values it is necessary to use to obtain accuracy to the nearest cent. *Ans.* \$32.50.

4. Same as exercise 3, except that the principal is \$14,625.

5. Same as exercise 3, except that the principal is \$146,250.

6. Verify the interest shown in the tables for \$1,000 at 1 per cent for 70 days at exact interest.

7. Find the ordinary simple interest on a note of \$1,250 bearing 7 per cent interest, dated March 10, 1921, and paid September 4, 1921, (*a*) by taking the exact number of days; (*b*) by finding the time in months and days and using 30 days to the month. Find also the exact interest on the same note. *Ans.* \$43.26, \$42.29, \$42.67.

8. Same as problem 7 for a note of \$7200 bearing 8 per cent interest dated January 1, 1920, and paid April 6, 1920. *Ans.* \$153.60, \$152.00, \$151.50.

9. In Illinois, paper falling due on Sunday or on a holiday is payable the next business day, interest being computed to the day of payment. If this rule is applied, find the ordinary and the exact interest on a 90 day note for \$1200 falling due on a July 3 falling on a Sunday, interest at 6 per cent.

7. Compound interest. When interest at the ends of certain periods of time is added to the principal, and the interest of each succeeding period is calculated on the principal formed by

thus adding interest, the total accumulated amount diminished by the original principal is called the **compound interest.**

The accumulated amount is sometimes called the compound amount.

8. Frequency of compounding. Although interest is almost always quoted at so much per annum, it is often made payable more frequently than once a year. That is, interest is converted into principal annually, semiannually, quarterly, or at some other regular interval. For this reason, we speak of interest **compounded** or **convertible** annually, semiannually, or quarterly. We speak also of a **conversion period** or **rest** by which we mean the time between two successive conversions of interest into principal. Thus, if interest at rate i is convertible quarterly, we say it is at rate i with quarterly rests.

When nothing is specified as to the conversion period, we shall throughout this book assume it to be one year.

9. Fundamental relations for compound interest.

Let P be the principal or present value,

S the amount to which P will accumulate,

n the number of conversion periods (number of years if no period is specified),

i the rate of interest for one period, (one year if no period is specified).

The amount of a principal P at the end of the first period is

$$P(1+i). \tag{6}$$

During the second period, we have $P(1+i)$ as a principal. The amount at the end of the second period is

$$P(1+i)^2.$$

Similarly, during the third period the principal is $P(1+i)^2$. Hence, the amount at the end of the third period is

$$P(1+i)^3.$$

Continuing this process, we note that the amount of P at the end of n periods is

$$S = P\,(1 + i)^n. \tag{7}$$

Formula (7) is the fundamental interest relation. It has been established for any positive integer n. When n is any positive fraction, the amount is defined as the result given by this relation. Indeed, as we shall see later, the relation is valid also for negative values of n, when we regard S as the value of P at a date n years in the past.

EXAMPLE 1. Find the amount of $1000 in 4 years at 5 per cent, compounded annually.

In this case, $S = \$1000(1.05)^4,$

$$= \$1215.51,$$

since from Table I, page 254, $(1.05)^4 = 1.21551$.

EXAMPLE 2. Find the amount of $1000 in four years at 5 per cent, compounded semiannually.

Here,

$$\begin{aligned} S &= \$1000(1.025)^8 \\ &= \$1000(1.21840) = \$1218.40, \end{aligned}$$

since from Table I, $(1.025)^8 = 1.21840$.

EXAMPLE 3. Find, without the use of Table I, the amount of $12,325 at 5 per cent for 36 years and 6 months, interest convertible semiannually. Use logarithms.*

FORM FOR SOLUTION:

$$\begin{aligned} S &= \$12325(1.025)^{73}. \\ \log 1.025 &= \\ \log\,(1.025)^{73} &= \\ \log 12325 &= \\ \hline \log S &= \\ S &= \end{aligned}$$

The fundamental relation (7) contains the four letters P,

* Those who do not understand the use of logarithms should at this point read Chapter XI.

i, n, and S. We can solve for any one of these in terms of the other three. Thus, from

$$S = P(1+i)^n,$$

by dividing by $(1+i)^n$, we have

$$P = \frac{S}{(1+i)^n} = Sv^n, \tag{8}$$

where $v = \dfrac{1}{1+i}$ is called the **discount factor.**

Next, to solve for i, divide the fundamental relation (7) by P, and thus obtain

$$(1+i)^n = \frac{S}{P}.$$

Extract the nth root of each member. This gives

$$1 + i = \sqrt[n]{\frac{S}{P}}.$$

Transpose and we have,

$$i = \sqrt[n]{\frac{S}{P}} - 1. \tag{9}$$

This formula can be used to find the rate i when the principal, the amount, and the time are known, but in solving numerical problems, it is usually better to apply logarithms to the members of (7) than to use (9).

Example 4. In 15 years, \$1000 at compound interest amounts to \$2078.93; find the rate. *Ans.* .05

Form for Solution:

From (7), $$S = P(1+i)^n.$$

$$\log S = \log P + n \log (1+i).$$

Transpose, $$n \log (1+i) = \log S - \log P. \tag{10}$$

$$\log (1+i) = \frac{\log S - \log P}{n}. \tag{11}$$

To use (10), fill in the form:

$$\begin{aligned}
\log (2078.93) &= \\
\log 1000 &= \\
\hline
n \log (1+i) &= \\
\log (1+i) &= \\
1+i &= \\
i &=
\end{aligned}$$

Finally, to solve (7) for n, we again take logarithms of each member of $S = P(1+i)^n$, or obtain n directly from (10). This gives

$$n = \frac{\log S - \log P}{\log (1+i)}. \tag{12}$$

EXAMPLE 5. In what time will \$1250 amount to \$9897.28 at 6 per cent, convertible semiannually. *Ans.* 35 years.

FORM FOR SOLUTION:

$$\begin{aligned}
\log 9897.28 &= \\
\log 1250 &= \\
\log S - \log P &= \\
\log (1+i) &= \\
\log (\log S - \log P) &= \\
\log [\log (1+i)] &= \\
\log n &= \\
n &=
\end{aligned}$$

10. Time required for money to double itself at compound interest at a given rate. If in (12) Art. 9, we take $S = 2P$, the amount is double the principal. Then,

$$n = \frac{\log 2P - \log P}{\log (1+i)} = \frac{\log 2}{\log (1+i)}. \tag{13}$$

It follows from Arts. 143 and 145 that

$$\log_{10} (1+i) = .4343 \left(i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \ldots\right).$$

EXAMPLE: Find the time required for money to double itself at 6 per cent interest, convertible annually. *Ans.* 11.89 years.

A convenient approximation to the time required for money to double itself may be obtained from (13) as follows:

$$\log_{10} 2 = .3010.$$

$$\log_{10}(1+i) = .4343\left(i - \frac{i^2}{2} + \frac{i^3}{3} - \cdots\right).$$

Hence, $$n = \frac{.3010}{.4343i\left(1 - \frac{i}{2} + \frac{i^2}{3} - \right)} = \frac{.693}{i} \cdot \frac{1}{1 - \frac{i}{2} + \frac{i^2}{3} -}$$

$$= \frac{.693}{i}\left(1 + \frac{i}{2}\right) \text{ approx.}$$

$$= \frac{.693}{i} + .35 \text{ approx.} \tag{14}$$

Thus, we note that an approximate rule for finding the time in which money will double itself may be stated as follows: To find the number of periods in which money will double itself, divide 69 by the rate per cent per period and add one third.

EXERCISES

1. Calculate the error in the result when the last example is carried out by this rule instead of using the more accurate relation (13).

2. Calculate the error in (14) when the rate of interest is, (1) one per cent, (2) two per cent, (3) eight per cent, (4) ten per cent.

11. Use of term " interest " to mean " compound interest." We shall be concerned so much in this work with compound interest and so rarely with simple interest that we shall from this point on employ the term interest to mean compound interest, unless we expressly state that simple interest is meant.

12. Computation of interest. For the calculation of interest, unless n be small, a table of logarithms, or a table of amounts of 1 (Table I, pp. 251–254) for various intervals of time n is practically indispensible because of the laborious character of the work when carried out without these aids.

It is also convenient at times to know how to find the interest on 1 for a time outside the range of the interest table without using logarithms. This can be done by making use of the

fact that the amount of 1 for $m + n$ years is equal to the amount for m years times that for n years; that is, we have

$$(1 + i)^{m+n} = (1 + i)^m (1 + i)^n.$$

For example, if our table runs to only 50 periods of time, the amount of 1 for 87 periods at the rate .05 would be

$$(1.05)^{87} = (1.05)^{50} \cdot (1.05)^{37},$$

and we can obtain the two factors of the second member from the table.

In many interest calculations, it is required to obtain results which are correct to the nearest cent. This means that with a principal of $20,000, it is, in general, necessary to use a seven place table of logarithms, to obtain the desired degree of accuracy. It is at once clear that a four-place or a five-place table of logarithms would not serve the purpose.

Exercise. Obtain as nearly as possible by the use of a four-place logarithmic table the amount of $24,625 at 5 per cent for 10 years, interest convertible semiannually. Find the error made with the four-place table by using Table I, p. 252, or a seven-place table of logarithms to obtain the higher degree of accuracy.

EXERCISES AND PROBLEMS

1. Verify by the use of logarithms the amount of 1 at 4 per cent for 25 years as given in Table I, p. 253.

2. Find the amount of 1 at 6 per cent for 100 years.

3. Find by the use of Table I, p. 254, the interest on $500 at 8 per cent compounded quarterly for 9 years 6 months.

4. By the use of (7) Art. 9, calculate the interest on $100,000 at 6 per cent per annum for ½ year. Calculate also the simple interest on the same principal for the same time. When formula (7) is applied, the result is sometimes called "true" interest. Which is the larger for the problem in hand, the simple interest or the "true" interest? How much larger? *Ans.* $2956.30. $3000.00.

5. Find the amount of $100,000 at 5 per cent per annum for 4 years 9 months,

(1) By calculating the compound interest for the entire time.

(2) By finding the compound amount at the end of the fourth year, and adding to this the simple interest on this amount for 9 months.

Which method gives the larger amount at the end of 4 years 9 months and how much larger?

6. Formula (7) Art. 9 gives the "true" interest whether n is integral or fractional, but it frequently happens in commercial transactions that compound interest is calculated to the end of the last whole period, and to this amount is added the simple interest on such amount for the remaining fraction of a period. Whom does such a practice favor, the creditor or the debtor, when judged by the results of problem 5?

7. Show that interest as given by formula (7) Art. 9 at any given rate i per annum for part of a year is less than simple interest for this time at rate i.

8. What is the amount of \$10,000 for 50 years at $3\frac{1}{2}$ per cent compounded annually?

9. In how many years will any sum double itself at 5, 6, 7, 8 per cent interest compounded annually? *Ans.* 14.21, 11.89, 10.24, 9.01.

10. In what time will \$100 amount to \$1000 at 5 per cent compounded annually. *Ans.* 47.19 years.

11. At what rate of interest will a sum double itself in 10 years?

12. One dollar is placed at interest at 4 per cent and allowed to accumulate at compound interest. During the same time another dollar is allowed to accumulate at 3 per cent. When will the first amount be twice the second? *Ans.* 71.74 years (7 place logarithms used).

13. Graphical representation of interest. Considerable insight may be gained into the accumulation of funds at interest by plotting the amount S of a given principal P corresponding to various assigned values of the time n. Thus, we may as in Fig. 1 use a horizontal scale to represent time, and a vertical scale to represent amounts. In Fig. 1, the distance from the base line to the horizontal line through P gives the principal, MQ is the amount of P at simple interest for 10 years, NQ is the simple interest on P for 10 years, MR is the amount of P at compound interest for 10 years, and NR is the compound interest on P for 10 years.

What does it signify that the graph for compound interest is above that for simple interest when the time n exceeds one year? What does it signify that the graph for compound interest is below that for simple interest when n is less than one year? (Fig. 2).

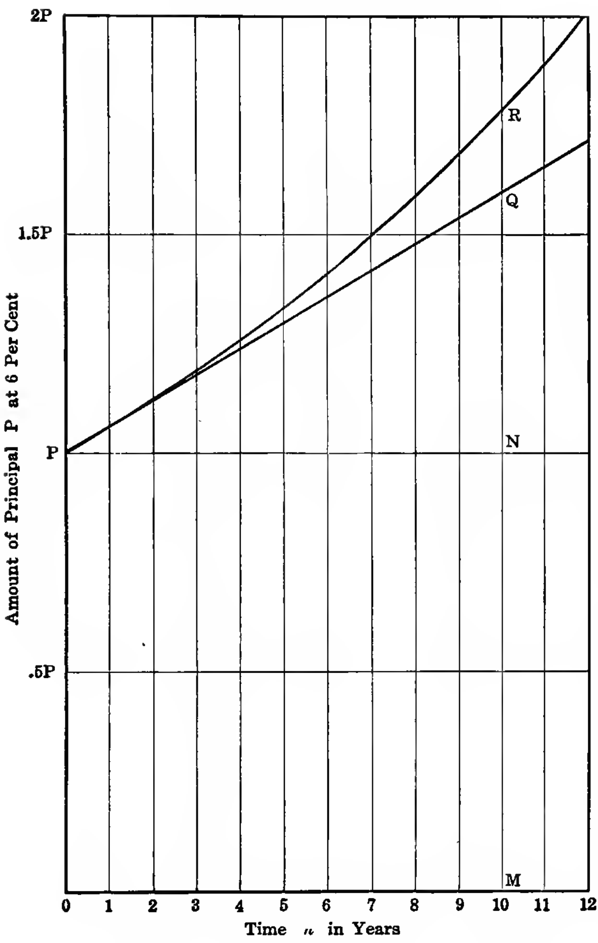

FIG. 1. Graphs of the amounts of a principal P at 6 per cent, (1) at simple interest (2) at compound interest.

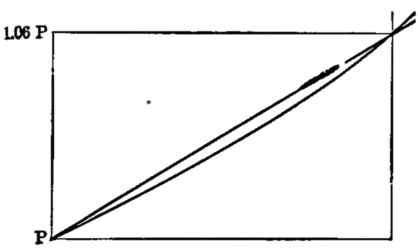

FIG. 2. A magnified portion of Fig. 1, showing the simple interest line and the compound interest curve for the first year.

The straight line for the graph of the amount of a principal at simple interest means that the amount increases uniformly with the time. It should be noted that at compound interest, the interest earned per unit of time increases as time increases.

EXERCISES

1. Plot graphs of the amounts of \$100 at 5 per cent, (1) simple interest, (2) compound interest.

2. Plot the graph of the present values of \$100 due in 0, 1, 2, 3, . . ., 15 years, using time along the horizontal scale and present values along the vertical. Take interest at 5 per cent and use Table II, page 258.

14. Limitations of simple interest. It should be observed that the lender who collects simple interest at the end of each year, and invests this interest at the same rate as the original principal has just the same accumulated amount of money at the end of any year as he would have had in case he had loaned his money in one transaction at compound interest. This fact suggests that simple interest carried far beyond a conversion period is not a rational kind of interest. If the borrower fails to pay his instalments of interest when due, he is depriving the lender of interest on these instalments.

Some may argue that compound interest is too exacting on the borrower. If interest to be paid is too exacting on the borrower, the equities cannot be adjusted by replacing compound interest by simple interest. The adjustment should be sought in a lowering of the rate of interest, and not in the adoption of such an absurd notion as that involved in using simple interest for a long period of time.

Capital at interest grows and the rate of growth is usually quoted as so much per period, but growth may well be thought of most naturally as something that is at least frequently and perhaps continuously changing. Indeed, the momently or continuous conversion of interest into principal treated in Art. 20 seems to be the most rational kind of interest.

Exercise. A party lends \$1000 at simple interest for 1 year at 6 per cent. At the end of this year, he lends both the principal and interest to

another party for 1 year at the same rate. At the end of the second year, he lends both the principal and interest for a third year at the same rate. Show that the resulting amount is the same as the amount of \$1000 for 3 years at 6 per cent convertible annually.

15. Nominal and effective rates of interest. The **effective rate of interest** is interest actually earned on 1 in a year. If interest is convertible more than once a year, the result is to give an effective rate greater than the **nominal** or quoted rate. Thus, we may have a nominal rate .04, but if interest is payable semiannually, the effective rate is

$$(1.02)^2 - 1 = .0404.$$

That is, on a principal of \$10,000, a nominal rate of .04 convertible semiannually gives in one year \$404 of interest.

16. A standard notation for nominal and effective rates. It is a common notation to use j for the nominal rate per annum, and to use m for the number of conversion periods per year.

Then $\frac{j}{m}$ is the interest on 1 for a conversion period. The amount of 1 at the end of a year is given by

$$\left(1 + \frac{j}{m}\right)^m,$$

and the interest earned is

$$\left(1 + \frac{j}{m}\right)^m - 1.$$

It is customary to use i for the effective rate for a year. Hence

$$i = \left(1 + \frac{j}{m}\right)^m - 1, \tag{15}$$

or, what seems perhaps more natural

$$1 + i = \left(1 + \frac{j}{m}\right)^m. \tag{16}$$

These relations (15) and (16) enable us to find i and $1 + i$ when we are given j and m.

From (16), we may also easily find j in terms of m and i as follows:

Extract the mth root of each member and transpose. This gives

$$1 + \frac{j}{m} = (1 + i)^{\frac{1}{m}}.$$

Solving for j, we have

$$j = m\left[(1 + i)^{\frac{1}{m}} - 1\right] \tag{17}$$

Sometimes the nominal rate j is written with a subscript to show the frequency of conversion in a year. Thus, $j_{(m)}$ means the nominal rate with m conversion periods in a year.

EXERCISES

1. The nominal rate of interest is 5 per cent and the interest is added to the principal each six months. What is the effective rate?

2. The effective rate of interest on a note is .04060401, find the corresponding nominal rate when interest is convertible quarterly.

3. Fill out the following table when $i = .06$. Give j to 4 significant figures.

m	j
1	
2	
4	
12	
52	
365	

4. Fill out the following table when $j = .06$. Give i to 4 significant figures.

m	i
1	
2	
4	
12	
52	
365	

17. The formulas of compound interest when $1 + i$ is replaced by $\left(1 + \frac{j}{m}\right)^m$.

The relations in Art. 9 become

$$S = P\left(1 + \frac{j}{m}\right)^{mn}, \tag{18}$$

and
$$P = \frac{S}{\left(1 + \frac{j}{m}\right)^{mn}} = Sv^n, \tag{19}$$

where
$$v = \frac{1}{\left(1 + \frac{j}{m}\right)^m}.$$

It is of considerable importance from the standpoint of simplicity to note that (18) and (19) are simply the formulas of Art. 9, where the rate per period of conversion is $\frac{j}{m}$ and the number of periods is mn, and where n is the number of years.

EXERCISE. Find the amount of a principal 1 at end of 1 year at a nominal rate of 6 per cent, convertible (1) twice a year, (2) four times a year, (3) twelve times a year, (4) 365 times a year or daily.

18. Present value. We often need to find the value of a sum of money at some time before it is due. By the **present value** of a sum S due in n years we mean the principal that will at a given rate amount to S in n years.

The problem of finding the present value is really solved by the relation (8) of Art. 9,

$$P = Sv^n, \tag{20}$$

where
$$v = \frac{1}{1+i}$$

will accumulate to 1 in one year.

The symbol v is often called the **discount factor.**

The values of powers of v are given in Table II, page 255.

EXERCISES

1. Find the present value of $10,000 due in 20 years if money is worth 4 per cent convertible semiannually. Use Table II. *Ans.* $4528.90.

2. Find the present value of $12,564 due in 20 years if money is worth 4 per cent convertible quarterly. *Ans.* $5667.85.

19. Discount. A consideration paid at the beginning of an interval of time for the use of capital during that interval is called **discount.**

If P is the principal and D the discount, then the borrower receives $P—D$ which may be considered as the present value of the P he is to pay at the end of the period. Hence, the discount may be defined as the difference between P due at some future time and its present value.

The **rate of discount** may be defined as the discount on 1 due in one year, and it is given by

$$\boldsymbol{d = 1 - v.} \tag{21}$$

Since $$v = \frac{1}{1 + i}, \text{ we have}$$

$$d = 1 - \frac{1}{1 + i} = \frac{i}{1 + i} = iv.$$

The simple relation

$$\boldsymbol{d = iv} \tag{22}$$

between the rate of interest and the rate of discount should be especially noted.

Exercise. Find the rate of discount, and the discount in a transaction under which a note of \$1000 bearing no interest, due in one year, is to be discounted so that the purchaser of the note will realize a rate of interest .06 on his money. *Ans.* $d = .05660$, $D = \$56.60$.

20. Interest convertible continuously. In Art. 16, we have seen that the amount of 1 at the end of one year with a nominal rate j convertible m times a year is

$$\left(1 + \frac{j}{m}\right)^m. \tag{23}$$

We have already, in the exercise of Art. 17 for $j = .06$, inquired into the value of expression (23) when m takes values 2, 4, 12, and 365 so as to make interest convertible semi-

annually, quarterly, monthly, and daily. It is both interesting and useful to investigate the value which $\left(1+\frac{j}{m}\right)^m$ takes when m is increased without bound. This is the case where interest is convertible continuously, or as is often said, momently.

For this purpose, we need to find the limiting value of

$$\left(1+\frac{j}{m}\right)^m$$

as m becomes infinite.

It follows from (3) Art. 142 that, as m increases without limit, $\left(1+\frac{j}{m}\right)^m$ approaches e^j, where $e = 2.71828+$ and is called the base of the natural or Naperian logarithms. More formally, we may write

$$\lim_{m \to \infty}\left(1+\frac{j}{m}\right)^m = \lim_{m \to \infty}\left[\left(1+\frac{j}{m}\right)^{\frac{m}{j}}\right]^j = e^j, \tag{24}$$

where "$m \to \infty$" stands for "m becomes infinite."

21. Effective rate with continuous conversion. Since the amount of 1 in one year at a nominal rate j with continuous conversion is

$$e^j,$$

the effective rate of interest is

$$i = e^j - 1. \tag{25}$$

When the nominal rate j with continuous conversion is given, we can calculate the effective rate i from (25).

Exercise. Given the nominal rate .06 convertible continuously; find the effective rate.

Form for Solution:

From (25), $1 + i = e^j.$

$\log_{10}(1+i) = j \log_{10} e.$

But $\log_{10} e = 0.43429$ (Art. 138, Chapter XI).

Then $\log_{10}(1+i) =$

$1 + i =$

$i =$

Complete the work.

22. Force of interest. When the effective rate i is given, and we assume momently conversion of interest, the corresponding nominal rate, given by j in the equality,

$$e^j = 1 + i$$

is usually denoted by δ and is called the **force of interest.** Thus,

$$\delta = \log_e (1 + i) \tag{26}$$

$$= \log_{10} (1 + i) \log_e 10 = 2.30259 \log_{10} (1 + i). \quad \text{(Art. 138.)}$$

EXERCISE. Find the force of interest that corresponds to a rate $i = .06$. *Ans.* $\delta = 0.05827$.

23. Amount of any principal in *n* years when interest is converted continuously. Since in this case, we may put

$$1 + i = e^j.$$

in the formula $S = P(1 + i)^n$, we have for continuous conversion of interest, the amount

$$S = Pe^{nj} = Pe^{n\delta}. \tag{27}$$

EXERCISES

1. Find the amount of \$1000 in 3 years at 5 per cent, convertible continuously. *Ans.* \$1161.83.

2. Find the amount of \$12525 in 5 years at 6 per cent convertible, (1) semiannually, (2) continuously.

24. The rationale and usefulness of continously convertible interest.

Let us consider a quantity S in which the rate of change or growth is at any instant of time n directly proportional to the quantity itself at that instant.

It is shown in calculus * that any such quantity obeys the law

$$S = Pe^{jn},$$

where P is the value of S when n equals zero.

* Townsend and Goodenough, *Essentials of Calculus*, p. 122.

This formula gives the law of growth which Lord Kelvin called the **compound interest law.** Now it seems rational that capital at interest should have a rate of growth at any time n, proportional to the quantity of capital at that time. This principle leads to the formula for the continuous conversion of interest. Any interval of conversion like a year is to a large extent an approximation to the real principle back of interest. The force of interest defined above occurs in the approximate solutions of certain problems of actuarial science * even when interest is converted annually.

25. Bank discount. When a merchant seeks payment on a bill by offering to sell it at a discount, the banker usually quotes him a rate of discount, not a rate of interest. Confusion has often arisen from the failure to keep this distinction in mind. Some writers have gone so far as to insinuate that it is dishonest to discount bills in accordance with bank practice and to say that when a banker discounts a bill or note of \$100 at 7 per cent for one year, he should deduct

$$d = \frac{\$7}{1.07} = \$6.54$$

and not \$7. This view would be correct if the bank were quoting the merchant 7 per cent interest, and meant to pay for the paper the present value of \$100 due in one year at 7 per cent interest. But if the banker says his rate of discount is 7 per cent, the merchant has no ground for complaint when the banker deducts \$7. The banker assigns the $d = .07$, and this statement should not be taken to mean that $i = .07$.

If the banker can employ his funds in discounting at 7 per cent, he is getting more than 7 per cent interest, and it is ignorance on the part of the merchant if he does not know this fact. The distinction between interest and discount consists in the way in which the rate is quoted.

The great majority of commercial bills have only a fractional

* See *Text-Book,* Institute of Actuaries, II, p. 171, (15).

part of a year to run, say 60 or 90 days. If that fraction is denoted by $\frac{1}{m}$, and d is the rate of discount per year, it is the usual practice for the banker to give for each unit of the bill an amount $1-\frac{d}{m}$ instead of

$$v^{\frac{1}{m}} = (1-d)^{\frac{1}{m}}.$$

given as the present value by Art. 18. In this way, the banker uses what is sometimes called **simple discount** instead of **compound discount.**

When simple discount is employed for intervals of time in excess of a year, it is likely to lead to absurd results. Thus, if a note had 10 years to run and were discounted at 10 per cent per year, simple discount, the value of the discounted note would be zero. If it were for more than 10 years, the note would have a negative value. In general, when n is large, $1-nd$ may have a negative value. Simple discount thus leads to erroneous and anomalous results.

The correct expression for the present value of 1 due in n years is

$$\boldsymbol{v^n = (1-d)^n}. \tag{28}$$

That is, just as we have compound interest, we have compound discount by repeating the discount operation.

To illustrate further, let us recall that when a bill purchased by the banker matures, he may employ the funds in discounting another bill. If d is the nominal rate of discount, and if the process of discounting is repeated m times a year, the present value of 1 due at the end of $\frac{1}{m}$th of a year is $1-\frac{d}{m}$, at the end of $\frac{2}{m}$th year is $\left(1-\frac{d}{m}\right)^2$, and so on. The present value of 1 due at the end of the year is

$$\boldsymbol{\left(1-\frac{d}{m}\right)^m}. \tag{29}$$

26. Force of discount. We may well inquire into the result of making m become infinite, and thus perform the operation of discounting continuously. When the discounting is thus performed, the nominal rate d may be written δ'. Thus, we have by Art. 142

$$v = \lim_{m \to \infty} \left(1 - \frac{d}{m}\right)^m = \lim_{m \to \infty} \left[\left(1 - \frac{d}{m}\right)^{-\frac{m}{d}}\right]^{-d} = e^{-\delta'}. \qquad (30)$$

In other words, we may say that, from Art. 142, the expression $\left(1 - \frac{d}{m}\right)^m$ approaches $e^{-\delta'}$ when m increases without limit.

The δ' in relation (30) is called the **force of discount** that corresponds to a discount factor v.

PROBLEMS

1. The price of thrift stamps is based on interest of 4 per cent compounded quarterly. What is the effective rate of interest?. *Ans.* 0.0406.

2. Compare the simple interest and compound interest on \$1000 for (*a*) 6 months at an effective rate .07; (*b*) 18 months, at same rate. *Ans.* (*a*) \$35.00, \$34.41; (*b*) \$105.00, \$106.82.

3. A money lender charges 3 per cent a month paid in advance for loans. What is the corresponding nominal rate of interest? What is the effective rate? *Ans.* 37.11 per cent; 44.13 per cent.

4. A wholesale hardware firm sells goods on 60 days' credit or 2 per cent off for cash. What is the highest rate of interest at which the retail merchant should under these circumstances borrow money so as to pay cash? *Ans.* 12.24 per cent.

5. What rate of interest is earned on money used in discounting bills at a discount rate of .08 per year?

6. What is the rate of discount at which a bank may as well employ its funds as to lend money at an interest rate of .07 per year?

7. A merchant gives his son permission to use his savings in discounting bills at 2 per cent off for cash on bills due in 90 days. What rate of interest can the son realize on money thus employed?

8. Find the force of discount when the discount factor is $v = 0.95$.

9. A merchant desires to obtain \$5000 from his banker for 90 days. If the rate of discount is 6 per cent, for what amount will the banker draw the note?

10. A life insurance company charges 5 per cent interest per annum in advance on policy loans with the privilege of repaying all or part at any time. A policyholder has a loan of \$1000. Three months prior to the date to which interest is paid he forwards the company a remittance for \$500 with instructions to apply on his policy loan. What would be the outstanding amount of the policy loan after giving proper credit?

27. Equated time or average due date. In settling a series of transactions, it frequently happens that it is desired to pay at one time the various sums due at different times. The **equated time** or **average due time** is the time at which they may all be paid with due regard to the equities involved.

Let the sums S_1, $S_2, \ldots\ldots$, S_p be due at the ends of n_1, n_2, $\ldots\ldots$, n_p years respectively, and let x be the equated time.

To find x on the hypothesis that money is worth a rate i we have by equating the present value of the total to the present value of the separate items, using

$$\frac{1}{1+i} = v,$$

$$v^x(S_1 + S_2 + \ldots + S_p) = v^{n_1}S_1 + v^{n_2}S_2 + \ldots + v^{n_p}S_p. \tag{31}$$

Taking logarithms of the members, and solving for x, we have

$$x = \frac{\log(v^{n_1}S_1 + v^{n_2}S_2 + \ldots + v^{n_p}S_p) - \log(S_1 + S_2 + \ldots + S_p)}{\log v} \tag{32}$$

$$= \frac{\log(S_1 + S_2 + \ldots + S_p) - \log(v^{n_1}S_1 + v^{n_2}S_2 + \ldots + v^{n_p}S_p)}{\log(1+i)}. \tag{33}$$

Exercise. Find the equated time for paying \$1000 due in five years and \$2000 due in ten years both without interest, if money is worth 6 per cent. *Ans.* 8.17 years.

28. Approximate method of finding equated time. The calculation of x from (33) Art. 27 presents no serious difficulty, but for certain practical purposes, particularly when the various times involved are short, the following approximate rule derived from (31) is employed:

The equated time for amounts due at different times, is given by multiplying each amount by the time to elapse before it becomes due, and then dividing the sum of these products by the sum of the amounts.

To obtain this rule from (31), replace v by $\dfrac{1}{1+i}$.

This gives

$$\frac{1}{(1+i)^x}(S_1+S_2+\dots+S_p) =$$

$$\frac{1}{(1+i)^{n_1}}S_1+\frac{1}{(1+i)^{n_2}}S_2+\dots+\frac{1}{(1+i)^{n_p}}S_p, \tag{34}$$

or

$$(1+i)^{-x}(S_1+S_2+\dots+S_p)$$

$$=(1+i)^{-n_1}S_1+(1+i)^{-n_2}S_2+\dots+(1+i)^{-n_p}S_p. \tag{35}$$

Expand $(1+i)^{-x}$, $(1+i)^{-n_1}$, $(1+i)^{-n_2}$, ..., $(1+i)^{-n_p}$ by the binomial theorem.* This gives

* The binomial theorem is used to expand a binomial. The student has used this theorem in elementary algebra for at least some special cases. Thus, he has used

$$(a+x)^2 = a^2+2ax+x^2,$$
$$(a+x)^3 = a^3+3a^2x+3ax^2+x^3,$$
$$(a+x)^4 = a^4+4a^3x+6a^2x^2+4ax^3+x^4,$$

and even higher powers.

For any positive integral exponent n, the binomial expansion is

$$(a+x)^n = a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3+$$
$$\dots+\frac{n(n-1)\dots(n-r+2)}{1\cdot 2\cdot 3\dots(r-1)}a^{n-r+1}x^{r-1}+\dots+x^n.$$

For a proof, see Rietz and Crathorne, *College Algebra*, Revised edition, pp. 93–94. When n is not a positive integer, the expansion does not terminate, but is a useful infinite series. When x is small compared to a, the first few terms of the expansion

$$(a+x)^n = a^n+na^{n-1}x+\frac{n(n-1)}{1\cdot 2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3+\dots$$

$$(1+i)^{-x} = 1 - xi + \frac{-x(-x-1)}{1 \cdot 2} i^2 - \ldots$$

$$(1+i)^{-n_1} = 1 - n_1 i + \frac{-n_1(-n_1-1)}{1 \cdot 2} i^2 - \ldots$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$(1+i)^{-n_p} = 1 - n_p i + \frac{-n_p(-n_p-1)i^2}{1 \cdot 2} - \ldots$$

Since powers of i above the first are relatively small, we neglect all terms which have such powers of i as a factor.

This gives from (35)

$$(1-xi)(S_1+S_2+\ldots+S_p) = S_1(1-n_1 i)+S_2(1-n_2 i)+\ldots + S_p(1-n_p i). \quad (36)$$

Solving (36) for x, and we have

$$x = \frac{n_1 S_1 + n_2 S_2 + \ldots + n_p S_p}{S_1 + S_2 + \ldots + S_p}. \quad (37)$$

The rule stated above is simply a verbal interpretation of this equation.

The difference in results between the approximate and the accurate method is simply that which arises from assuming simple discount in the one case and compound discount in the other. For short durations, therefore, no serious error is involved in using the approximate method and it has the advantage of simplicity. Since in most commercial transactions simple interest and simple discount are used for periods less than a year, the approximate method for equated time may very properly be used when the due dates of all sums involved are less than a year in advance.

The approximation (37) gives a value in excess * of the correct value given by (33) except in the trivial case in which $n_1 = n_2 = \ldots = n_p$.

give a valuable approximation to the exact value. The condition that x be small compared to a is satisfied by $(1 + i)^n$ in interest problems since i is usually small compared to 1.

* For proof, see *Text-Book*, Institute of Actuaries, Part I, 1915, pp. 25–26.

29. Equation of value. The expression **equation of value** is used to denote any equation that gives a statement of equality, as of a given date, among sums due at different dates. Thus, if the given date is the present, then for \$1000 , due in two years and \$100 due in one year, interest 6 per cent, we have

$$984.34 = \frac{100}{1.06} + \frac{1000}{(1.06)^2},$$

or \$984.34 due now, is the equivalent of \$100 due in 1 year and \$1000 due in two years if money is worth 6 per cent. If the given date is one year from now, the equation of value is

$$1043.40 = 100.00 + \frac{1000}{1.06},$$

or \$1043.40 due one year hence is the equivalent of \$100 due one year hence and \$1000 due two years hence if money is worth 6 per cent.

Along with the cases of present values already treated, and the case treated under equated time, Art. 27, there is the important case of finding what unknown sum S can be paid at some given time n years from now to discharge debts $S_1, S_2, \ldots, S_p$ due in $n_1, n_2, \ldots, n_p$ years, respectively. We get the equation of value by finding the value of each item at any convenient date. The resulting value S turns out to be the same no matter to what date the various sums are accumulated or discounted. Thus if the present time is taken, we have

$$Sv^n = S_1v^{n_1} + S_2v^{n_2} + \ldots + S_pv^{n_p}, \tag{38}$$

in which S is the unknown.

If a date t years in the future is taken, we have

$$Sv^{n-t} = S_1v^{n_1-t} + S_2v^{n_2-t} + \ldots + S_pv^{n_p-t}. \tag{39}$$

Since v^{-t} is a factor of both members of (39) S does not depend upon t.

EXERCISES

1. Write the equation of value in the notation above as of a date, (1) one year from now, (2) as of n years from now.

2. Assuming money to be worth 5 per cent per year, find the sum required to discharge now two debts one of which is \$1000 due in 10 years

without interest, and the other is $2000 due in four years without interest. *Ans.* $2259.32.

3. Find the sum required to discharge the two debts in exercise 2 at a date two years hence.

4. Two debts of $1250 and $700 are due without interest in two years and eighteen months, respectively. A third debt of $900 with interest at 7 per cent is due in one year. Find the sum required to discharge all three debts six months hence.

MISCELLANEOUS PROBLEMS

1. Find the ordinary and the exact interest on $1256.36 for 87 days at 7 per cent. *Ans.* $21.25. and $20.96.

2. What nominal rate of interest compounded semiannually is the equivalent of a discount rate of 6 per cent for one year? *Ans.* 0.0628.

3. What rate of interest compounded quarterly is equivalent to 6 per cent compounded semiannually?

4. A wholesale dealer's terms are "cash in ninety days or two per cent off for cash in thirty days." What rate of discount can a merchant afford to pay if he borrows money to take advantage of the rebate?

5. Fill out the following table to four significant figures where δ means force of interest and δ' force of discount.

i	d	δ	δ'	
.02				
.04				
.06				
.08				
.10				

6. The Text-Book of the Institute of Actuaries gives $\frac{i+d}{2}$ as an approximate value for δ. From the table in problem 5 find the errors of the approximation.

7. If the $24 paid for Manhattan Island in 1614 had been at interest at 6 per cent convertible annually to the present date, what would be the amount?

8. Find the interest on a debt of $14275 for 4 years 3 months and 5 days at 6 per cent (1) with semiannual rests, (2) with quarterly rests.

9. Show that the interest calculated from the amount given by (7) Art. 9 at 6 per cent for ½ year is less than the simple interest at 6 per cent for the same period.

10. Find the present value of a debt that will amount to $3235 in 5 years, if money is worth 6 per cent. *Ans.* $2417.38.

11. Find the present value of $6250 due in 5 years and 6 months without interest, when money is worth 6 per cent.

12. The sum of the amount of 1 in 2 years at a certain nominal rate of interest convertible half-yearly, and of the present value of 1 due 2 years hence at the same nominal rate of discount convertible half-yearly, is 2.00480032. Find the rate.

(*Text-Book*, Institute of Actuaries, Part I, p. 21).

13. Find the equated time for paying four bills of $500, $1000, $1500, $2000 due in 3 months, 6 months, 9 months, and 1 year hence, respectively, when money is worth 6 per cent per annum, using (1) the approximate method, (2) the exact method.

14. A banker discounts a note for 3 mos. at 6 per cent. What effective rate of interest is earned? What would be the equivalent nominal rate of interest compounded semiannually? *Ans.* .0623 and .0614.

15. Same as problem 14 except that the time is 6 mos. instead of 3 mos. *Ans.* .0628 and .0618.

16. Find the sum required 8 years from now to discharge one sum of $1250 due in 12 years without interest and another of $2500 due in 6 years but bearing interest at the rate of 4 per cent per annum, when money is worth 6 per cent.

17. How long must $1000 be left to accumulate at 6 per cent convertible semiannually to amount to twice as much as $2000 left for the same time at 3 per cent convertible semiannually?

18. Find to the nearest cent the amount of $1000 for 1 year at a nominal rate of 6 per cent, convertible (1) semiannually, (2) quarterly, (3) monthly, (4) daily, (5) continuously.

19. A capitalist gives a university \$1,000,000 in 5 per cent bonds interest payable semiannually to be held in trust and to become available when the accumulated amount is \$1,500,000. If the same rate of interest is earned on the interest as on the face of the bonds, find the time required for the gift to become available.

20. In what time will \$1000 amount to \$2000 at a nominal rate of 6 per-cent convertible; (1) annually, (2) semiannually, (3) quarterly, (4) monthly (5) daily, (6) continuously.

21. Give a graphical representation of problem 12, Art. 12.

CHAPTER II

ANNUITIES CERTAIN

30. Annuities. An **annuity** is a succession of periodical payments. Strictly speaking the word annuity implies a yearly payment, but it is now generally understood that the word applies to all periodical payments, whether made annually, quarterly, monthly, biennially, or otherwise. However, when the interval is a fractional part of a year the annuity is usually measured by the total payment made during the year. This total payment is called the **annual rent.** Income from rented property, insurance premiums, and pensions are examples of annuities.

31. Annuities certain. Annuities may last for a fixed term of years or for a period of time depending upon some contingency, for example the life of an individual. When the annuity is to continue during a fixed period it is called an **annuity certain.**

32. Amount of an annuity certain. Unless otherwise indicated the first payment of an annuity is supposed to be made at the end of the first period. The sum to which the various payments accumulate is usually called the **amount of the annuity,** although **accumulation** of the annuity expresses the idea better. In the mathematical discussion of annuities, it is usual to treat first the case in which the annual rent is 1.

The symbol $s_{\overline{n|}}$ is universally used to represent the amount of an annuity of 1 per annum payable annually for n years. If the effective rate of interest i is used, then $s_{\overline{n|}}$ will depend upon n and i. The first payment of 1 made at the end of the first year will be at interest for $n - 1$ years and will accumulate to

$$(1 + i)^{n-1}.$$

The second payment will be at interest for $n-2$ years and will accumulate to $(1+i)^{n-2}$, and so on. The last payment will be a cash payment of 1 at the end of the period. We have then

$$s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + \ldots + (1+i)^2 + (1+i) + 1$$
$$= 1 + (1+i) + (1+i)^2 + \ldots + (1+i)^{n-1}.$$

This is a geometrical progression * of n terms in which the first term is 1 and the common ratio is $1+i$.

The sum of this finite series is then $\dfrac{(1+i)^n - 1}{i}$. Hence,

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \tag{1}$$

gives the amount of an annuity of 1 in terms of n and i.

If the annual rent is R instead of 1, and if K represents the amount, we have

$$K = Rs_{\overline{n}|} = R\,\frac{(1+i)^n - 1}{i}. \tag{2}$$

Equations (1) and (2) contain three variables i, n and $s_{\overline{n}|}$. If any two of these are known the third is fixed in value. To find n if i and $s_{\overline{n}|}$ are given, we solve (2) for n as follows: Write (2) in the form

$$(1+i)^n = \frac{Ki}{R} + 1 = \frac{Ki + R}{R}.$$

Taking logarithms of both sides and solving for n we have

$$n = \frac{\log (Ki + R) - \log R}{\log (1+i)}, \tag{3}$$

if K, R and i are such numbers that n is an integer. If n is not an integer this formula gives an approximate value of n. See problem 4, page 34.

To find i if n and $s_{\overline{n}|}$ are given, it is necessary to solve an

* A discussion of progressions will be found in Chapter XII. For geometrical progressions see Arts. 150–153.

equation of degree $n - 1$. The solution of such equations is discussed in Art. 36.

In the above, it was assumed that we knew the effective rate i. Ordinarily we know the nominal rate j and the number of conversion intervals m per annum. We can then find i from the formula,

$$(1 + i) = \left(1 + \frac{j}{m}\right)^m,$$

or we may express $s_{\overline{n}|}$ and n in terms of j and m by replacing $(1 + i)$ by $\left(1 + \frac{j}{m}\right)^m$ in formulas (1), (2), (3), obtaining

$$s_{\overline{n}|} = \frac{\left(1 + \frac{j}{m}\right)^{mn} - 1}{\left(1 + \frac{j}{m}\right)^m - 1}, \tag{4}$$

$$Rs_{\overline{n}|} = R\frac{\left(1 + \frac{j}{m}\right)^{mn} - 1}{\left(1 + \frac{j}{m}\right)^m - 1}, \tag{5}$$

$$n = \frac{\log\left[s_{\overline{n}|}\left(1 + \frac{j}{m}\right)^m - s_{\overline{n}|} + 1\right]}{\log\left(1 + \frac{j}{m}\right)^m}. \tag{6}$$

EXERCISES AND PROBLEMS

1. A man sets aside \$200 at the end of each year towards a fund for his son's college expenses. If he invests the money at 4 per cent effective, what will be the amount at the end of 10 years?

Solution: From (2) we have

$$200 . s_{\overline{10}|} = 200\,\frac{(1.04)^{10} - 1}{.04}.$$

Performing the arithmetical operations we find the result \$2401.22. Since the interest rate here is at 4 per cent we may find $s_{\overline{10}|}$ from Table III to be 12.0061, from which we easily obtain the result.

2. If in problem 1, the interest realized had been 4.2 per cent what would have been the amount? *Ans.* \$2423.60.

3. If in problem 1, the time had been 20 years and only \$100 had been set aside annually, what would have been the amount?

4. If money can be invested at 4 per cent effective, how many full years will be necessary to accumulate a fund of at least \$3000 from \$200 set aside at the end of each year.

SOLUTION: This is an application of (3). We have $i = 04$, $R = \$200$, $K = \$3000$.

$$n = \frac{\log\left(.04\cdot\dfrac{3000}{200} + 1\right)}{\log(1.04)} = \frac{\log(1.60)}{\log(1.04)} = \frac{0.204120}{0.017033} = 11.98 \text{ years.}$$

Hence 12 years are necessary, but the amount will be a little over \$3000.

5. Work problem 4, if the interest is 3 per cent.

6. A man invests \$150 at the end of each year. At the end of 8 years he has accumulated \$1357.75. Write out the equation whose solution will give the rate of interest. Find the answer as nearly as possible from the $s_{\overline{n|}}$ table.

7. If \$1500 invested at the end of each year for two years amounts to \$3082.50, what is the rate of interest?

8. If \$1000 invested at the end of each year for three years amounts to \$3199.20, what is the rate of interest?

9. If in problem 1, the rate had been 4 per cent nominal, with two conversion intervals per year, what would have been the amount?

SOLUTION: We have $j = .04$, $m = 2$, and $n = 10$, and (5) becomes

$$200\, s_{\overline{10|}} = 200\,\frac{(1.02)^{20}-1}{(1.02)^2-1}.$$

From Table I we find $(1.02)^{20} = 1.4859474$, $(1.02)^2 = 1.0404$.

Hence

$$200\, s_{\overline{10|}} = 200\frac{.4859474}{.0404} = 2405.68.$$

10. Work problem 3, with interest at 4 per cent nominal converted semiannually.

11. Find the amount of an annuity of \$1000 for 8 years (*a*) with effective rate 6 per cent; (*b*) with nominal 6 per cent, converted quarterly. *Ans.* \$9897.47; \$9946.03.

12. Work problem 1 if the money is set aside at the beginning of each year.

33. Amount of an annuity payable p times a year. If an annuity of 1 per annum is payable in p equal instalments at equal intervals during the year, the amount at the end of n years is represented by the symbol,

$$s_{\overline{n}|}^{(p)}.$$

If the interest at nominal rate j is convertible p times a year the problem of finding $s_{\overline{n}|}^{(p)}$ reduces to the problem of finding the amount of an annuity of $\frac{1}{p}$ for np periods at a rate $\frac{j}{p}$ per period. For this case we have

$$s_{\overline{n}|}^{(p)} = \frac{1}{p}\, s_{\overline{np}|}, \tag{7}$$

$$\left(s_{\overline{np}|} \text{ to be computed at rate } \frac{j}{p}\right).$$

For example, an annuity of 1 per year paid in quarterly instalments for 10 years at 6 per cent, would be

$$s_{\overline{10}|}^{(4)} = \frac{1}{4}\, s_{\overline{40}|} \text{ computed at rate } .015,$$

provided interest is converted four times a year.

If interest is converted yearly and i is the effective rate, $s_{\overline{n}|}^{(p)}$ can be expressed in terms of n, p and i as follows. At the end of the pth part of a year, $\frac{1}{p}$ is paid. This sum will remain at interest for $n - \frac{1}{p}$ years and will amount to

$$\frac{1}{p}(1+i)^{n-\frac{1}{p}}.$$

In like manner the second instalment of $\frac{1}{p}$ will amount to $\frac{1}{p}(1+i)^{n-\frac{2}{p}}$, and so on for np instalments, the last of which

will be paid at the end of n years. Beginning with the last payment, we will then have

$$s_{\overline{n}|}^{(p)} = \frac{1}{p} + \frac{1}{p}(1+i)^{\frac{1}{p}} + \frac{1}{p}(1+i)^{\frac{2}{p}} + \dots + \frac{1}{p}(1+i)^{n-\frac{2}{p}} + \frac{1}{p}(1+i)^{n-\frac{1}{p}}.$$

This is a geometrical progression of np terms, with $\frac{1}{p}$ for the first term, and $(1+i)^{\frac{1}{p}}$ as the ratio. Its sum is (Art. 152)

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{p[(1+i)^{\frac{1}{p}} - 1]}. \tag{8}$$

If the rate of interest is not unusual, this formula may be made easier for computation by writing

$$s_{\overline{n}|}^{(p)} = \frac{i}{j_{(p)}} \cdot \frac{(1+i)^n - 1}{i} = \frac{i}{j_{(p)}} \cdot s_{\overline{n}|} = s_{\overline{1}|}^{(p)} \cdot s_{\overline{n}|} \tag{9}$$

where $\qquad j_{(p)} = p[(1+i)^{\frac{1}{p}} - 1]$. (See Art. 16).

For ordinary rates of interest $s_{\overline{n}|}$ can be found from Table III, pp. 259–262, and $\frac{i}{j_{(p)}} = s_{\overline{1}|}^{(p)}$ from Table VIII, p. 271.

If interest is converted m times a year and j is the nominal rate, the above series would be written

$$s_{\overline{n}|}^{(p)} = \frac{1}{p} + \frac{1}{p}\left(1+\frac{j}{m}\right)^{\frac{m}{p}} + \frac{1}{p}\left(1+\frac{j}{m}\right)^{\frac{2m}{p}} + \dots + \frac{1}{p}\left(1+\frac{j}{m}\right)^{m(n-\frac{1}{p})},$$

in which $(1+i)$ is replaced by $\left(1+\frac{j}{m}\right)^m$.

The sum is

$$s_{\overline{n}|}^{(p)} = \frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}} - 1\right]}. \tag{10}$$

If $p = m$, (10) reduces to (7).

If instead of 1, the annual rent is R and the amount K, we find from (8), (9), and (10),

$$K = Rs_{\overline{n}|}^{(p)} = R\,\frac{(1+i)^n - 1}{p[(1+i)^{\frac{1}{p}} - 1]}, \tag{11}$$

$$K = Rs_{\overline{n}|}^{(p)} = R\,\frac{i}{j_{(p)}} \cdot s_{\overline{n}|} = Rs_{\overline{1}|}^{(p)} \cdot s_{\overline{n}|}, \tag{12}$$

$$K = Rs_{\overline{n}|}^{(p)} = R\,\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{p\left[\left(1+\frac{j}{m}\right)^{\frac{m}{p}} - 1\right]}. \tag{13}$$

If K, i, and p are known, (11) may be solved for n giving

$$n = \frac{\log\left\{\frac{Kp}{R}[(1+i)^{\frac{1}{p}} - 1] + 1\right\}}{\log(1+i)}, \tag{14}$$

or

$$n = \frac{\log\left\{\frac{K}{R}\,j_{(p)} + 1\right\}}{\log(1+i)}. \tag{15}$$

EXERCISES AND PROBLEMS

1. Find the amount of an annuity of \$1200 per year paid in four quarterly instalments of \$300 for 7 years if the interest rate is 5 per cent effective.

SOLUTION: Here we have $R = 1200$, $i = .05$, $p = 4$, $n = 7$,

and (11) becomes $K = 1200\,\dfrac{(1.05)^7 - 1}{4[(1.05)^{\frac{1}{4}} - 1]}$.

From Table I, $(1.05)^7 = 1.4071004$.

From Table VII, $4[(1.05)^{\frac{1}{4}} - 1] = .0490889$.

Hence $K = 1200\,\dfrac{.4071004}{.0490889} = \$9951.75$.

2. A man deposits \$100 in a 4 per cent savings bank at the end of every three months. If interest is converted semiannually what amount will be to his credit at the end of 6 years?

3. How long will it take to accumulate \$1000 by depositing \$10 at the end of each month in a savings bank paying 3 per cent effective? Give answer to the nearest month.

SOLUTION: Here $R = 120$, $K = 1000$, $p = 12$ and formula (14) gives

$$n = \frac{\log \{100\,[(1.03)^{\frac{1}{12}} - 1] + 1\}}{\log (1.03)}.$$

By logarithms, we find $(1.03)^{\frac{1}{12}} = 1.00247$.

Then $$n = \frac{\log 1.247}{\log (1.03)} = \frac{.09587}{.01284} = 7.48 \text{ years,}$$

or $$n = 7 \text{ years, } 6 \text{ months.}$$

Since the interest rate is not unusual, we may use (15), finding from Table VII, $j_{(p)} = .0295952$.

Hence $$n = \frac{\log \left\{\frac{1000}{120}\;.0295952 + 1\right\}}{\log(1.03)} = 7.46 \text{ years,}$$

or $$n = 7 \text{ years } 6 \text{ months.}$$

4. Work problem 3 if the interest is 4 per cent.

5. Fill out the following table for the amount of an annuity of \$100 per year for ten years, interest at 4 per cent.

	Interest convertible		
The Annuity Payable	Annually	Semiannually	Quarterly
Annually			
Semiannually			
Quarterly			

34. Present value of an annuity certain. The present value of an annuity is the sum of the present values of all the future payments. The present value of an annuity certain of 1 per annum is represented by the symbol $a_{\overline{n}|}$. The present values of the different payments (Art. 18), beginning with the first payment are

$$(1 + i)^{-1}, (1 + i)^{-2}, \ldots, (1 + i)^{-n}, \text{ or } v, v^2, \ldots v^n.$$

We have then for the present value of the annuity certain

$$a_{\overline{n}|} = v + v^2 + v^3 + \ldots + v^n,$$

a geometrical progression whose first term is v, last term v^n, and common ratio v. Summing this finite series, we find

$$a_{\overline{n}|} = \frac{v - v^{n+1}}{1 - v}.$$

Dividing both numerator and denominator by v, and remembering that $v = (1 + i)^{-1}$ we have

$$a_{\overline{n}|} = \frac{1 - v^n}{i}, \tag{16}$$

giving the present value of an annuity certain of 1 per annum in terms of n and the effective rate i.

EXERCISE. Derive the formula

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

by multiplying each member of

$$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}$$

by $(1 + i)^{-n}$.

For an annuity certain of R per annum, if we represent the present value by A, we have

$$A = Ra_{\overline{n}|} = \frac{R(1 - v^n)}{i}. \tag{17}$$

This relation (17) contains the variables i, n, A and R. It is possible to solve for any one of these when the others are given. Thus, writing (17) in the form

$$v^n = (1 + i)^{-n} = \frac{R - Ai}{R}, \text{ or } (1 + i)^n = \frac{R}{R - Ai},$$

and taking logarithms of each member, we have

$$n \log (1 + i) = \log R - \log (R - Ai),$$

or

$$n = \frac{\log R - \log (R - Ai)}{\log (1 + i)}, \tag{18}$$

which gives the term of the annuity when the present value, the annual rent and the effective rate are known. In this expression R, A and i should be such numbers that n is an integer. If n is not an integer the formula is approximately true.

If the nominal rate j and the number of conversion intervals m are given, substitution of $\left(1 + \frac{j}{m}\right)^m$ for $(1 + i)$ in (17) and (18) gives

$$A = \frac{R\left[1 - \left(1 + \frac{j}{m}\right)^{-mn}\right]}{\left(1 + \frac{j}{m}\right)^m - 1}, \tag{19}$$

and

$$n = \frac{\log R - \log \left\{R - A\left[\left(1 + \frac{j}{m}\right)^m - 1\right]\right\}}{m \log \left(1 + \frac{j}{m}\right)}. \tag{20}$$

We shall consider in Art. 36 the problem of solving (17) for i when n, R, and A are given.

EXERCISES AND PROBLEMS

1. A man pays \$52.17 paving tax at the end of each year for 10 years. If the interest charge is 5 per cent what is the actual tax for the paving.

SOLUTION: To find the present value A, of an annuity certain of \$52.17 per annum for 10 years, we have

$$A = 52.17 \cdot \frac{1 - (1.05)^{-10}}{.05} = 52.17 \cdot 7.7217 = 402.84.$$

$$\text{Tax} = \$402.84.$$

2. Find the present value of an annuity of \$1000 per annum for 5 years if money is worth 4 per cent effective. *Ans.* \$4451.82.

3. A house for sale is listed at \$10,000. The seller agrees to take \$4000 cash and \$1000 per annum for 6 years without interest. If money is worth 6 per cent effective in such transactions, what reduction was made in the price of the house? *Ans.* \$1082.68.

4. Find the present value of an annuity of \$1000 per annum for 5 years if money is worth 4 per cent, interest converted quarterly.

5. The present value of an annuity of \$1250 per annum is \$8079. If money is worth 5 per cent effective, what is the term?

6. The present value of an annuity of \$875 per annum for 3 years is \$2475.03. Write down an equation of the third degree whose solution is the effective rate. By substitution, show that .03 will satisfy the equation.

35. Present value of an annuity payable for n years in p instalments per year. If an annuity of 1 per annum is payable in p equal instalments at equal intervals during the year, the present value is represented by the symbol

$$a_{\overline{n}|}^{(p)}.$$

If the interest at nominal rate j is convertible p times a year, the problem of finding the present value reduces to the problem of finding the present value of an annuity of $\frac{1}{p}$ per period for np periods at a rate $\frac{j}{p}$ per period. That is,

$$a_{\overline{n}|}^{(p)} = \frac{1}{p}\, a_{\overline{np}|} \text{ (computed at rate } \frac{j}{p}). \tag{21}$$

If interest is converted yearly, $a_{\overline{n}|}^{(p)}$ can be expressed in terms of n, p and i. The first payment of $\frac{1}{p}$ will be made at the end of $\frac{1}{p}$ years. Its present value is

$$\frac{1}{p}(1+i)^{-\frac{1}{p}} \text{ or } \frac{1}{p}\, v^{\frac{1}{p}}.$$

The second payment will be made at the end of $\frac{2}{p}$ years, and

its present value is $\frac{1}{p}\,v^{\frac{2}{p}}$, and so on. We find then

$$a_{\overline{n}|}^{(p)} = \frac{1}{p}\,(v^{\frac{1}{p}} + v^{\frac{2}{p}} + v^{\frac{3}{p}} + \ldots + v^{n-\frac{1}{p}} + v^{n}).$$

The part in parenthesis is a geometrical progression of np terms with first term $v^{\frac{1}{p}}$ and ratio $v^{\frac{1}{p}}$. Its sum is

$$\frac{v^{n+\frac{1}{p}} - v^{\frac{1}{p}}}{v^{\frac{1}{p}} - 1} = \frac{v^{\frac{1}{p}}\,(v^{n} - 1)}{v^{\frac{1}{p}}\,(1 - v^{-\frac{1}{p}})} = \frac{1 - v^{n}}{v^{-\frac{1}{p}} - 1} = \frac{1 - (1+i)^{-n}}{(1+i)^{\frac{1}{p}} - 1}.$$

Hence,
$$a_{\overline{n}|}^{(p)} = \frac{1 - (1+i)^{-n}}{p[(1+i)^{\frac{1}{p}} - 1]}, \tag{22}$$

and
$$A = Ra_{\overline{n}|}^{(p)} = R\,\frac{1 - (1+i)^{-n}}{p[(1+i)^{\frac{1}{p}} - 1]}, \tag{23}$$

If both numerator and denominator be multiplied by i, we may write

$$Ra_{\overline{n}|}^{(p)} = R\,\frac{1 - (1+i)^{-n}}{i} \cdot \frac{i}{p[(1+i)^{\frac{1}{p}} - 1]} = Ra_{\overline{n}|} \cdot \frac{i}{j_{(p)}},$$

and hence
$$A = Ra_{\overline{n}|}\,\frac{i}{j_{(p)}} = Ra_{\overline{n}|}s_{\overline{1}|}^{(p)}. \tag{24}$$

The most common values of p are 2, 4 and 12. For these values of p, tables of $\frac{i}{j_{(p)}}$ have been constructed for the usual interest rates, thus making formula (24) well adapted to computation (See Table VIII). The equations (23) and (24) involve the variables n, A, R, p, and i. The problems that arise in this connection may require the solution for any one of these letters when the others are given.

To find n if A, R, p and i are given, write equation (23) in the form

$$\frac{R - Ap\,[(1+i)^{\frac{1}{p}} - 1]}{R} = (1+i)^{-n},$$

or
$$(1+i)^{n} = \frac{R}{R - Ap\,[(1+i)^{\frac{1}{p}} - 1]}.$$

Taking logarithms, we have

$$n \log (1+i) = \log R - \log\left\{R - Ap\left[(1+i)^{\frac{1}{p}} - 1\right]\right\},$$

$$n = \frac{\log R - \log\left\{R - Ap\left[(1+i)^{\frac{1}{p}} - 1\right]\right\}}{\log (1+i)},$$

$$= \frac{\log R - \log\left\{R - A \cdot j_{(p)}\right\}}{\log (1+i)}. \tag{25}$$

If, instead of the effective rate i, we have given the nominal rate j and the number of conversions intervals per annum, m, we replace $1 + i$ by $\left(1 + \frac{j}{m}\right)^m$ in (23) and (25), and obtain

$$A = \frac{R}{p} \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1}, \tag{26}$$

whence

$$n = \frac{\log R - \log\left\{R - Ap\left[\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1\right]\right\}}{m \log\left(1 + \frac{j}{m}\right)}. \tag{27}$$

EXERCISES AND PROBLEMS

1. Find the present value of an annuity of $1200 per annum in monthly instalments for 7 years if money is worth 5 per cent effective.

SOLUTION: $$A = \$1200 \cdot a_{\overline{7}|} \cdot \frac{i}{j_{(12)}}.$$

From Table IV, $a_{\overline{7}|} = 5.7863734$. From Table VIII, $\frac{i}{j_{(12)}} = 1.0227147$.

Hence, $A = 1200 \cdot 5.7863734 \cdot 1.0227147 = \$7101.37.$

2. Find the present value of an annuity of $1200 per annum in (*a*) semi-annual instalments, (*b*) quarterly instalments, for 7 years, if money is worth 5 per cent effective.

3. A house is sold for $2000 cash, and $50 per month for 7 years, without interest. If money is worth 6 per cent effective what would be the cash price of the house. *Ans.* $5440.57.

4. Find the present value of an annuity of $860 per annum in quarterly instalments for 8 years if money is worth 5 per cent nominal, converted quarterly. *Ans.* $5648.54.

5. Find the present value of an annuity of $860 per annum in quarterly instalments for 8 years if money is worth 5 per cent nominal, converted semiannually. *Ans.* $5648.51.

6. How long will it take a man to pay for a piece of property priced $6000, if he pays $1000 down and $800 at the end of each year until full payment is made. Money worth 6 per cent.

7. From first principles, derive a formula for the accumulated value of 1 per month for n months at an effective rate of i per annum.

8. From first principles, derive a formula for the present value of 1 per month for n months at an effective rate of i per annum.

36. To find the rate. We have seen in Art. 32 that the problem of finding the rate of interest, when n and $s_{\overline{n}|}$, or n and $a_{\overline{n}|}$ are given, reduces usually to the solution of an equation of high degree. Fortunately in most annuity problems, we can find roughly approximate answers quite easily. Starting with such approximate rates, by the use of annuity tables such as Tables III and IV, there is a method of successive approximation which enables us to find the rate to any desired degree of accuracy. This method can be explained best by numerical examples.

When both the present value $a_{\overline{n}|}$ and the amount $s_{\overline{n}|}$ of an annuity are given the interest rate is easily found by formula (40) Art. 41. (See problem 5, Art 41).

EXAMPLE 1. An annuity of 1 per annum amounted to 8 dollars in 7 years. What was the rate of interest?

SOLUTION: Here $s_{\overline{n}|} = 8$ and $n = 7$, and formula (1) becomes

$$8 = \frac{(1+i)^7 - 1}{i}.$$

or $$(1+i)^7 - 8i - 1 = 0. \qquad (1)$$

From Table III, for $n = 7$ and $i = .04$ we find

$$s_{\overline{n}|} = 7.8982945,$$

while for $n = 7$ and $i = .045$, we find

$$s_{\overline{n}|} = 8.0191518.$$

The rate i is then between .04 and .045.

By interpolation by proportional parts, we can get a first approximation to the rate corresponding to $s_{\overline{7|}} = 8$. In this interpolation it is unnecessary to use all the decimal places in the tabular values of $s_{\overline{7|}}$. Thus for $i = .04$ and $i = .045$ we may use 7.898 and 8.019 respectively. The difference is .121. The i corresponding to $s_{\overline{7|}} = 8.000$ is then

$$.04 + \frac{102}{121} .005 = .0442.$$

For many purposes this result is sufficiently accurate.

To obtain a closer approximation, we may write

$$i = .0442 + h,$$

where h is a very small number. Substituting this value for i in equation (1), we have

$$(1.0442 + h)^7 - 8(.0442 + h) - 1 = 0. \tag{2}$$

If we expand the binomial $(1.0442 + h)^7$, we have

$$(1.0442)^7 + 7(1.0442)^6 h + 21\,(1.0442)^5 h^2 + \ldots + h^7,$$

in which h is so small that the terms in h^2, h^3, ..., h^7 may be dropped. Equation (2) then becomes

$$(1.0442)^7 + 7(1.0442)^6\, h - 8h - 1.3536 = 0,$$

or,
$$1.353586 + 9.074029h - 8h - 1.3536 = 0,$$

or,
$$1.074029h = .000014,$$

which to five decimal places gives

$$h = .00001,$$

and a second approximation to i is .04421.

Starting again with

$$i = .04421 + h,$$

we may repeat the process and obtain i to any required degree of accuracy.

Example 2. The present value of an annuity of \$1250 per annum for 18 years is \$16262. What is the rate of interest?

Solution: The present value of an annuity of 1 per annum for 18 years at the same rate is

$$\frac{16262}{1250} = 13.0096.$$

From Table IV, for $i = .035$, $n = 18$, we find

$$a_{\overline{n|}} = 13.1896817,$$

while for $i = .04$ we find

$$a_{\overline{n}|} = 12.6592970.$$

Hence i lies between .035 and .04. By interpolation by proportional parts we find as a first approximation, $i = .0367$. To obtain a second approximation, substitute

$$i = .0367 + h$$

in equation (16) Art. 34, which then becomes

$$13.0096 = \frac{1 - (1.0367 + h)^{-18}}{.0367 + h}, \tag{1}$$

or, $$(1.0367 + h)^{-18} + 13.0096\ (.0367 + h) - 1 = 0. \tag{2}$$

Expanding $(1.0367 + h)^{-18}$ by the binomial theorem and neglecting powers of h higher than the first, we have

$$(1.0367)^{-18} - 18\ (1.0367)^{-19}\, h,$$

to replace $(1.0367 + h)^{-18}$ in equation (2).
By logarithms $(1.0367)^{-18} = 0.52269$, $(1.0367)^{-19} = 0.50419$, and equation (2) reduces to

$$0.52269 - 18\ (0.50419)h + 13.0096\ (0.0367 + h) - 1 = 0,$$

or $$3.93418h = -\ 0.00014.$$

Hence $$h = -\ 0.00004,$$

and a second approximation for i is

$$0.0367 - 0.00004 = 0.03666.$$

EXERCISES

1. Check the work under example 1, page 45, by making the substitution $i = .0442 - h$.

2. Check the work under example 2, page 45, by making the substitution $i = .0367 - h$.

3. The present value of an annuity of 1 per annum for 11 years is 9. What is the rate of interest?. *Ans.* .03503.

4. An annuity of \$375 per annum for 10 years amounts to \$4912.50. Find the rate of interest.

5. By interpolation by proportional parts from Table I, find first approximations to the values of $(1.0325)^{24}$, and $(1.049)^{17}$.

6. By interpolation by proportional parts find from Table II first approximations to $(1.021)^{-30}$, $(1.047)^{-18}$, and $(1.059)^{-35}$.

37. Deferred annuities. **A deferred annuity** is an annuity whose term is to begin at the end of an assigned number of years. The amount of an annuity of 1 per annum payable for n years deferred m years is represented by the symbols

$$_{m|}s_{\overline{n|}}, \quad _{m|}s^{(p)}_{\overline{n|}}$$

according as the payments are made once or p times a year. The amount is evidently the same as the amount of an ordinary annuity for the same term. In symbols, we may then write

$$_{m|}s_{\overline{n|}} = s_{\overline{n|}},$$
$$_{m|}s^{(p)}_{\overline{n|}} = s^{(p)}_{\overline{n|}}.$$

The present value of an annuity of 1 deferred m years is represented by

$$_{m|}a_{\overline{n|}}, \quad _{m|}a^{(p)}_{\overline{n|}}.$$

according as the payments are made once or p times a year. Such a deferred annuity may be thought of as an annuity received for $m + n$ years, combined with an annuity paid by the receiver for the first m years. The present value of an annuity for n years deferred m years is then the difference between the present values of the ordinary annuity for $m + n$ years and that for m years. That is, for an annuity of 1 per annum deferred m years, we have then in symbols

$$_{m|}a_{\overline{n|}} = a_{\overline{m+n|}} - a_{\overline{m|}}, \tag{29}$$
$$_{m|}a^{(p)}_{\overline{n|}} = a^{(p)}_{\overline{m+n|}} - a^{(p)}_{\overline{m|}}.$$

EXERCISES AND PROBLEMS

1. Find the present value of an annuity of \$750 per annum for 9 years (*a*) deferred 5 years, (*b*) deferred 10 years, if money is worth 4 per cent effective. *Ans.* \$4583.48; \$3767.28.

2. Using the methods of Arts. 34, 35, derive formulas for $_{m|}a_{\overline{n|}}$, $_{m|}a^{(p)}_{\overline{n|}}$.

3. Find the present value of an annuity of \$600 per annum paid in quarterly instalments for 12 years, deferred 4 years if money is worth 5 per cent effective.

38. Annuities due. We have been considering annuities whose payments were made at the end of each interval. The payments may be made at other times. If they are made at the beginning of the interval the annuity is called an **annuity due.** The amount and the present value of an annuity certain due are represented by the symbols

$$\mathbf{s}_{\overline{n}|},\ \mathbf{s}^{(p)}_{\overline{n}|},\ \mathbf{a}_{\overline{n}|},\ \mathbf{a}^{(p)}_{\overline{n}|},$$

in black roman type. In considering the amount of an annuity due, it is customary in the United States to allow the payments to accumulate for the full term of n years, that is for one period after the last payment has been made. In the case of annual payments, the first payment of 1 will then be at interest for n years, the second for $(n - 1)$ years and so on. We have then

$$\begin{aligned} \mathbf{s}_{\overline{n}|} &= (1 + i)^n + (1 + i)^{n-1} + \dots + (1 + i), \\ &= (1 + i)\,[(1 + i)^{n-1} + (1 + i)^{n-2} + \dots + 1]. \end{aligned}$$

But the series in the bracket is the expression found for $s_{\overline{n}|}$ (Art. 32). Hence

$$\mathbf{s}_{\overline{n}|} = (1 + i)\, s_{\overline{n}|}. \tag{30}$$

In a similar manner we can show that

$$\mathbf{s}^{(p)}_{\overline{n}|} = (1 + i)^{\frac{1}{p}}\, s^{(p)}_{\overline{n}|}. \tag{31}$$

An annuity due for n years may be considered as made up of one cash payment and an ordinary annuity for $n - 1$ years. We have then for the present value of an annuity due of 1 per annum

$$\mathbf{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}. \tag{32}$$

If the annuity is payable p times a year, we have

$$\mathbf{a}^{(p)}_{\overline{n}|} = (1 + i)^{\frac{1}{p}}\, a^{(p)}_{\overline{n}|} = (1 + i)^{\frac{1}{p}}\, a_{\overline{n}|}\,\frac{i}{j_{(p)}}. \tag{33}$$

EXERCISES AND PROBLEMS

1. A man aged 22 pays \$23.27 a year on a 20 payment life insurance policy of \$1000, first payment to be made at the beginning of the first year. If he should die at the end of 10 years just before the eleventh year premium is due, how much would his estate be increased by having taken the insurance instead of having put the premiums into a savings bank paying 3½ per cent interest effective? *Ans.* \$717.46.

2. A man deposits \$300 at the beginning of each quarter in a savings bank. At the end of 5 years, what amount stands to his credit? The bank pays 4 per cent nominal converted semiannually.

3. Find an expression for the amount of an annuity due at the time of the last payment, (*a*) when the payments are annual; (*b*) when the payments are *p* times a year.

4. Find the present value of the premium of \$23.27 mentioned in problem 1 under the assumption that the man will live to pay the 20 premiums.

5. A man buying a house agrees to pay \$500 down and \$500 every six months until he has paid \$7,000. If money is worth 6 per cent, what should be the cash price of the house.

6. Show that $\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1$.

39. Tables. We have seen that a great deal of the arithmetical work connected with interest and annuity problems may be avoided by the use of tables giving the values of $(1+i)^n$, v^n, $a_{\overline{n}|}$ and $s_{\overline{n}|}$. The construction of such tables requires much computation and great care in checking results. Much of the work can however be avoided by using what is known as the continuous process in table construction. We can illustrate the method in connection with the construction of a table for v^n. The method of direct calculation of each tabular value for given values of i and n involves the checking of each individual result. To insure accuracy it is almost necessary to have the work done by two independent computers. If however we first calculate v we can find v^2 by multiplying the first tabular value by v. Multiplying the second tabular value by v gives v^3 and so on to the end of the table. If we check the final value by direct calculation by logarithms, we have then checked the whole table. The table may also be computed by calculating the last value first, then, by repeated multiplications by $1 + i$, the other tabular values can be found and checked by an in-

dependent calculation of the first value, v. In practice it is well to check now and then before the end of the table is reached.

The basic number to be calculated should be computed to several more decimal places than are to be finally retained in the table.

In the above construction it should be noted that three steps were necessary.

1. One tabular value, either the first or the last, was computed.

2. A working formula or rule was found connecting two consecutive tabular values, that is, a formula connecting the nth tabular value with either the $(n-1)$th or the $(n+1)$th value. Thus,

$$v^n = v \cdot v^{n-1},$$

or

$$v^n = (1+i)\, v^{n+1}.$$

For example,

$$v^{49} = v \cdot v^{48}, \text{ or } v^{49} = (1+i)\, v^{50}.$$

3. The last computed value was checked by independent computation.

Three such steps are necessary in any table construction by the continuous process. The second step, that of finding a workable formula connecting two consecutive tabular values, is usually the most difficult, and in many tables impossible. However, for the important functions $s_{\overline{n}|}$ and $a_{\overline{n}|}$, such a formula can easily be found. To find $s_{\overline{n+1}|}$ from $s_{\overline{n}|}$ we proceed as follows:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}.$$

$$s_{\overline{n+1}|} = \frac{(1+i)^{n+1} - 1}{i}.$$

Multiplying both members of the first by $1+i$, we have

$$\begin{aligned}(1+i)\, s_{\overline{n}|} &= \frac{(1+i)^{n+1} - (1+i)}{i},\\ &= \frac{(1+i)^{n+1} - 1}{i} - 1,\\ &= s_{\overline{n+1}|} - 1,\end{aligned}$$

or $$s_{\overline{n+1}|} = (1 + i)\, s_{\overline{n}|} + 1. \tag{34}$$

Starting then with $n = 1$ to build up a 5 per cent table of $s_{\overline{n}|}$ we have

$$s_{\overline{1}|} = 1.0000000,$$

$$s_{\overline{2}|} = (1.05)\,(1.0000000) + 1 = 2.0500000,$$

$$s_{\overline{3}|} = (1.05)\,(2.0500000) + 1 = 3.1525000,$$

and so on to the end of the table, perhaps checking every tenth value.

A working formula for $a_{\overline{n}|}$ is found in a similar way.

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i},$$

$$a_{\overline{n-1}|} = \frac{1 - (1 + i)^{-(n-1)}}{i}.$$

Multiplying both members of the first equation by $1 + i$, we have

$$(1 + i)\, a_{\overline{n}|} = \frac{1 + i - (1 + i)^{-(n-1)}}{i}.$$

$$= \frac{1 - (1 + i)^{-(n-1)}}{i} + 1.$$

$$= a_{\overline{n-1}|} + 1.$$

or $$a_{\overline{n-1}|} = (1 + i)\, a_{\overline{n}|} - 1. \tag{35}$$

To construct this table we compute the last value, say for $n = 50$. If it is a 5 per cent table we find

$$a_{\overline{50}|} = 18.2559255,$$

then $$a_{\overline{49}|} = (1.05)(18.2559255) - 1 = 18.1687217,$$

$$a_{\overline{48}|} = (1.05)(18.1687217) - 1 = 18.0771578,$$

and so on to the value $a_{\overline{1}|}$.

EXERCISES

1. Find by the continued process a four place table giving the first ten values of $(1+i)^n$ for $i = .07$. Check the last result by logarithms.

2. Derive a working formula giving $a_{\overline{n+1}|}$ in terms of $a_{\overline{n}|}$, and compare its practicability as a working formula for computing a table for $a_{\overline{n}|}$ with the formula given above.

3. Show how values for v^n may be checked from a table for $a_{\overline{n}|}$.

4. Construct a $4\frac{1}{4}$ per cent table for the first ten values of $a_{\overline{n}|}$, correct to four places of decimals.

5. Show how values of v^n beyond the limits of a given table may be easily computed.

6. Prove the formula $a_{\overline{m+n}|} = v^m a_{\overline{n}|} + a_{\overline{m}|}$, and show how the formula may be used to find present values beyond the limit of a given table.

40. The annuity which 1 will purchase. The annuity which may be purchased for 1 is the annuity whose present value is 1. If R is the annual rent of such an annuity, then its present value is $Ra_{\overline{n}|}$, [Art. 34] and R may be found from the equation

$$Ra_{\overline{n}|} = 1.$$

Thus, we find

$$R = \frac{1}{a_{\overline{n}|}} = \frac{i}{1-(1+i)^{-n}},$$

or

$$\boldsymbol{R = \frac{1}{a_{\overline{n}|}} = \frac{i}{1-v^n}}. \tag{36}$$

If the purchase price is A, then evidently the annual rent will be

$$RA = \frac{1}{a_{\overline{n}|}}\ A.$$

If the annuity is payable in p instalments a year we have

$$R = \frac{1}{a_{\overline{n}|}^{(p)}} = \frac{p[(1+i)^{\frac{1}{p}}-1]}{1-(1+i)^{-n}} = \frac{j_{(p)}}{i} \cdot \frac{1}{a_{\overline{n}|}}. \tag{37}$$

Table V gives values of $\frac{1}{a_{\overline{n}|}}$ for ordinary rates of interest. From

Table VII we can find $j_{(p)}$ for $p = 2, 4, 12$ and thus easily find R from equation (37).

41. The annuity which will amount to 1. The amount of an annuity of R dollars per annum for n years is (Arts. 32, 33) $Rs_{\overline{n}|}$ or $Rs_{\overline{n}|}^{(p)}$ according as the payments are annual or in p instalments per annum. If the amount is 1, we have, respectively

$$\boldsymbol{R = \frac{1}{s_{\overline{n}|}} = \frac{i}{(1+i)^n - 1},} \tag{38}$$

$$R = \frac{1}{s_{\overline{n}|}^{(p)}} = \frac{p[(1+i)^{\frac{1}{p}} - 1]}{(1+i)^n - 1} = \frac{j_{(p)}}{i} \cdot \frac{1}{s_{\overline{n}|}}. \tag{39}$$

If the amount is K, the annual rent will be RK. The value of R can easily be computed for ordinary interest rates from a table for $\frac{1}{s_{\overline{n}|}}$. Such tables are however not necessary if we have a table for $\frac{1}{a_{\overline{n}|}}$. The value of $\frac{1}{s_{\overline{n}|}}$ is found by simply subtracting i from the corresponding value of $\frac{1}{a_{\overline{n}|}}$. This is easily shown as follows:

$$\frac{1}{a_{\overline{n}|}} - \frac{1}{s_{\overline{n}|}} = \frac{i}{1-(1+i)^{-n}} - \frac{i}{(1+i)^n - 1} =$$

$$i\left[\frac{(1+i)^n}{(1+i)^n - 1} - \frac{1}{(1+i)^n - 1}\right] = i\left[\frac{(1+i)^n - 1}{(1+i)^n - 1}\right] = i,$$

or

$$\boldsymbol{\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i.} \tag{40}$$

If instead of the effective rate i, the nominal rate j and number of conversion intervals m are given, the corresponding expressions for R are easily obtained by the substitution of

$$(1+i) = \left(1 + \frac{j}{m}\right)^m$$

in formulas (36), (37), (38) and (39).

EXERCISES AND PROBLEMS

1. A fraternity chapter builds a house costing \$35,000 and arranges to pay \$5000 down and the remainder principal and interest in 20 equal annual payments. If the interest is 6 per cent effective what is the annual payment?

HINT: The payments form an annuity certain whose present value is \$30,000.

2. A house priced \$5000 is sold on the basis of equal monthly payments to extend through 7 years. If money is worth 6 per cent effective, what is the monthly payment?

3. What sum must be set aside yearly to provide for the payment of a debt of \$2000 due in 5 years, if the payments may be invested at 4 per cent effective? *Ans.* \$369.25 at end of each year.

4. A city borrows \$100,000 for ten years to provide a drainage system. What amount must be set aside yearly from the taxes into a sinking fund to provide for the payment of the loan when due? The sinking fund may be invested at 4½ per cent. *Ans.* \$8137.88.

5. The present value of an annuity of \$1000 per annum is \$12085.32 and the amount of this annuity would be \$30539.00. Find the interest rate.

42. Perpetuities. There are annuities whose payments continue indefinitely. For example, \$100 deposited in a bank paying 4 per cent effective will yield \$4 per year for an indefinite period. The interest payments may be considered as a perpetual annuity. Such an annuity is called a **perpetuity.**

The amount of such an annuity of course increases indefinitely, but the present value is definite and in the case of an annuity of 1 per annum is represented by the symbol, a_∞, which may be considered as the limit of $a_{\overline{n|}}$ as n increases indefinitely. If n increases indefinitely, v^n approaches zero as a limit and we may write

$$a_\infty = \text{limit}\, a_{\overline{n|}} = \text{limit}\, \frac{1 - v^n}{i} = \frac{1 - 0}{i} = \frac{1}{i}. \qquad (41)$$

This result could have been obtained by simply reasoning out the answer to the question: What amount of money placed at interest at rate i will produce 1 per annum?

The present value of a perpetuity whose first payment is made now, a perpetuity due, is

$$\mathbf{a}_{\infty} = 1 + a_{\infty} = 1 + \frac{1}{i} = \frac{1+i}{i} = \frac{1}{d}. \tag{42}$$

There are perpetuities whose payments are made at intervals of more than a year. For example, a city pavement with a probable life of 20 years, has just been laid. What sum could be set aside and allowed to accumulate to provide for an indefinite number of renewals, assuming the cost remains constant? The problem is the problem of finding the present value of a perpetuity whose payments are made every 20 years.

Let r be the interval between payments of 1 measured in years. The present value of the first payment which is made at the end of r years is v^r, of the second payment v^{2r} and so on. The present value of the perpetuity, represented by the symbol $a_{\infty,r}$, is then

$$a_{\infty,r} = v^r + v^{2r} + v^{3r} + \ldots .$$

This is an infinite geometrical progression whose first term is v^r and whose common ratio is v^r. Its sum (see Art. 153) is then

$$\frac{v^r}{1-v^r} = \frac{1}{v^{-r}-1} = \frac{1}{(1+i)^r-1} = \frac{1}{i}\cdot\frac{i}{(1+i)^r-1} = \frac{1}{i}\cdot\frac{1}{s_{\overline{r}|}}.$$

Hence $$a_{\infty,r} = \frac{1}{i}\cdot\frac{1}{s_{\overline{r}|}}. \tag{43}$$

If the payments are R instead of 1, the present value is

$$R \quad a_{\infty,r}.$$

To find $Ra_{\infty,r}$ for ordinary rates of interest, we simply multiply $\frac{1}{s_{\overline{r}|}}$ taken from table by $\frac{R}{i}$. For other rates of interest, $\frac{1}{(1+i)^r-1}$ must be calculated and the result multiplied by R.

If in the foregoing pavement problem we add to the original cost of the pavement the present value of the cost of the renewals, we have the total present cost of a pavement which will

last indefinitely. This is called the **capitalized cost** of the pavement. In general, the capitalized cost of anything is the sum of its original cost and the present value of indefinite renewals.

PROBLEMS

1. A section of city pavement costs \$48,000. Its life is 20 years. Find the capitalized cost, if money is worth $3\frac{1}{2}$ per cent effective. *Ans.* \$96495.

2. In his will a church member leaves \$500 for painting and repairing the church, and sufficient funds to provide for a like amount to be spent every 4 years for the same purpose. If money is worth 4 per cent what was the total amount of the bequest? *Ans.* \$3,443.26.

3. Is it cheaper to use ordinary shingles costing 7 dollars per thousand and lasting 12 years than to use asphalt shingles costing 9 dollars per thousand and lasting 20 years, assuming that it costs 1 dollar per thousand more to lay asphalt shingles than to lay ordinary shingles, and assuming that money is worth 6 per cent effective? On what does the answer depend?

4. What is the capitalized cost of an automobile truck whose original cost is \$1600? Its useful life is 4 years, at the end of which time it has a second-hand value of \$500. Money worth 5 per cent.

43. Continuous annuities. In Arts. 20–24 we treated the continuous conversion of interest by assuming that interest is momently added to the principal. Similarly, we may regard an annuity of annual rent 1 for which the payments are made momently. Such an annuity is called a **continuous annuity.**

We found in Art. 33 that the accumulation in n years on an annuity of annual rent 1, payable in p instalments per year, is

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{p\left[(1+i)^{\frac{1}{p}} - 1\right]}.$$

The accumulation of the continuous annuity for n years is equal to the limit approached by $s_{\overline{n}|}^{(p)}$ as p increases without bound. That is, the continuous annuity denoted by $\bar{s}_{\overline{n}|}$ is given by

$$\bar{s}_{\overline{n}|} = \lim_{p \to \infty} \frac{(1+i)^n - 1.}{p\left[(1+i)^{\frac{1}{p}} - 1\right]}. \tag{44}$$

To evaluate this limit, we first use the binomial theorem to expand $(1+i)^{\frac{1}{p}}$. This gives

$$(1+i)^{\frac{1}{p}} = 1 + \frac{1}{p}i + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)}{2!}i^2 + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)\left(\frac{1}{p}-2\right)}{3!}i^3 + \dots$$

Then we have

$$\lim_{p\to\infty} p\left[(1+i)^{\frac{1}{p}}-1\right] = \lim_{p\to\infty} p\left[\left(1+\frac{1}{p}i+\frac{\frac{1}{p}\left(\frac{1}{p}-1\right)}{2!}i^2 + \frac{\frac{1}{p}\left(\frac{1}{p}-1\right)\left(\frac{1}{p}-2\right)}{3!}i^3+\dots\right)-1\right]$$

$$= i - \frac{i^2}{2} + \frac{i^3}{3} - \dots, \tag{45}$$

$$= \log_e(1+i), \tag{46}$$

the series in (45) being the logarithmic series treated in Art. 143. In Art. 22, we found

$$\log_e(1+i) = \delta, \tag{47}$$

the force of interest. Hence, the limiting value of the expression in (44) is

$$\bar{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{\delta}. \tag{48}$$

$$= \frac{i}{\delta} s_{\overline{n|}}. \tag{49}$$

In a similar manner, we obtain from the value of $a_{\overline{n|}}^{(p)}$ the present value of the continuous annuity,

$$\bar{a}_{\overline{n|}} = \frac{1-v^n}{\delta} = \frac{i}{\delta} a_{\overline{n|}}. \tag{50}$$

In problems involving annuities paid in a considerable number of instalments per year, continuous annuities may be conven-

iently used to obtain approximate results. For example, the present value of an annuity of 1 per year paid in weekly instalments for 5 years with interest at 5 per cent is found from the formula, (see (22) Art. 35)

$$a_{\overline{5}|}^{(52)} = \frac{1-(1.05)^{-5}}{52[(1.05)^{\frac{1}{52}}-1]},$$

to be 4.435. The present value of a continuous annuity of 1 per annum for the same time and rate is, from equation (50), 4.437. The latter is easier to calculate and is a satisfactory approximation for many purposes. Further use of continuous annuities will be found in the exercises which follow.

The following table of values of $\delta = \log_e(1+i)$ and of $\frac{i}{\delta}$ for common values of i will facilitate such computations.

i	δ	i/δ
.025	0.0246926	1.01245
.03	0.0295588	1.01493
.035	0.0344014	1.01740
.04	0.0392207	1.01987
.045	0.0440169	1.02233
.05	0.0487902	1.02480
.06	0.0582689	1.02971
.07	0.0676587	1.03460
.08	0.0769611	1.03949

EXERCISES

1. Find the amount of a continuous annuity of annual rent \$120 for 5 years at an effective rate of 6 per cent. Verify the value of δ for $i = .06$ in the above table.

SOLUTION:

$$\bar{s}_{\overline{5}|} = \frac{.06}{\log_e(1.06)}\, s_{\overline{5}|}.$$

From Table III, $s_{\overline{5}|} = 5.6370930.$ $\dfrac{.06}{\log_e(1.06)} = 1.02971.$

Hence $\bar{s}_{\overline{5}|} = 1.02971 \cdot 5.637093 = 5.80457.$

and $120 \cdot \bar{s}_{\overline{5}|} = \$696.55.$

$$\log_e(1.06) = 2.3025851 \cdot \log_{10}(1.06),$$
$$= (2.3025851)(0.02530587) = 0.0582689 = \delta.$$

2. Find the amount of an annuity of annual rent \$120 payable in monthly instalments for 5 years at an effective rate of 6 per cent and compare the result with that of exercise 1.

3. Work exercise 2 for weekly instalments.

4. Find the present value of a continuous annuity of annual rent \$1000 for 5 years at an effective rate of 4 per cent. *Ans.* \$4540.28.

5. Find the accumulated value at the end of 5 years of a continuous annuity of annual rent \$1000 at a nominal rate of 4 per cent convertible momently. HINT: From Art. 21, $1 + i = e^j$.

6. Find approximately the present value of an annuity of 1 dollar a day for 10 years counting 365 days to the year. Interest 5 per cent effective.

7. Find approximately the accumulated value of an annuity of 1 dollar a day for 10 years, counting 365 days to the year. Interest 5 per cent effective.

MISCELLANEOUS PROBLEMS

1. Two men deposit \$300 a year in a savings bank paying 4 per cent effective. One makes his deposit of \$300 at the end of each year. The other deposits \$150 at the end of each half-year. At the end of 10 years just after making deposits, how much more does the latter have on deposit than the former? *Ans.* \$35.67.

2. A man deposits \$300 at the end of each year in a bank paying 4 per cent effective. Another man deposits \$300 at a bank paying 4 per cent nominal with half yearly conversion intervals. At the end of 10 years how much more does the latter have on deposit than the former? *Ans.* \$6.69.

3. How long will it take savings of 1 dollar a month to amount to $100 if deposited in a bank paying 3 per cent nominal converted semiannually? *Ans.* In 90 months the amount is slightly more than $100.

4. According to the terms of an endowment policy, the insured is to receive an income of $1000 at the end of each year for 15 years. He wishes to change this to an income for 10 years. At $3\frac{1}{2}$ per cent effective what income will he receive?

5. Find n in terms of $s_{\overline{n}|}$ and i, and in terms of $a_{\overline{n}|}$ and i. Equate the two values of n and prove $\dfrac{1}{a_{\overline{n}|}} - \dfrac{1}{s_{\overline{n}|}} = i.$

6. A perpetuity of $1000 per year is changed to an annuity due for 20 years. What is the annual rent if money is worth 4%? *Ans.* $1768.79.

7. A man leaves a perpetual annuity of annual rent $1,000 to be equally divided between two hospitals. One is to receive the full annuity until it has received its share, after which the other will receive a perpetuity. How long does the first hospital receive the annuity if money is worth 5 per cent effective? Does the rate of interest influence the result?

8. How long will it take to pay for a piece of property worth $4000 by paying $50 at the beginning of each month if money is worth 6 per cent effective?

9. A invests $100 and receives $6 a year dividends for 15 years when the business fails and he loses the principal. B invests $100 and receives no dividends for 15 years, then $6 a year for 5 years when he sells his investment for $110. Which man made the better investment if money was worth 5 per cent?

10. Money invested in a rubber plantation earned no dividends for eight years. Beginning with the ninth year a regular return of 50 per cent per annum began. If this rate continues indefinitely, what rate of interest does the investor really receive on his investment if money in general is worth 4 per cent. *Ans.* 37 per cent.

11. Prove $\dfrac{1}{a_{\overline{n}|}^{(p)}} - \dfrac{1}{s_{\overline{n}|}^{(p)}} = j_{(p)}.$

12. To assure a college education for his son a father invests at 4 per cent at his son's birth a sum sufficient to provide $100 per month for 5 years, the first $100 to be paid at the end of the first month of the nineteenth year. What was the amount invested?

13. An insurance policy maturing, the policyholder is given the option of a cash payment of $10,000 or an annuity certain for 10 years. At $3\frac{1}{2}$ per cent, what is the annual rent?

14. If in problem 13, the policyholder had been given an option of an annuity certain due for 10 payments, what would have been the annual rent?

15. Find the present value of an annuity due of $1000 paid in semiannual instalments for 10 years if money is worth 4 per cent effective.

16. Show that the value of a perpetuity due payable quarterly is

$$\frac{1}{4\left\{1 - (1 - d)^{\frac{1}{4}}\right\}}.$$

17. A policyholder, at the time of the maturing of a policy for $10,000, is given an option of an annuity due of $1,173.60 for 10 years. What interest rate is used by the company?

18. Find $a_{\overline{n}|}$ in terms of a_{∞} and n.

19. An annuity of $1,000, payable in semiannual instalments for 20 years is purchased for $14,500. What rate of interest was realized by the purchaser?

20. What amount of money set aside at the end of each month will amount to $100 in 100 months if money is worth 5 per cent effective?

21. A construction company is requested to furnish bids for the construction of a factory building for a manufacturing concern. According to the bids the construction company will build the structure for $300,000 in cash or for 10 equal half yearly instalments of $40,000 without interest, the first instalment being payable at once. What is the present value of the difference between the two propositions to the manufacturing company if it can borrow at 6 per cent payable semiannually?

22. A small hospital will cost $50,000 to build and $10,000 per year payable at the end of each year for maintenance. Assuming that it must be rebuilt at the same cost at the end of each 40 years, what sum at 5 per cent compounded yearly will provide for this hospital permanently?

23. A loan of $10,000 is to be repaid in 10 years by uniform semiannual payments which include interest at 6 per cent payable half-yearly for the first 5 years of the time, and at 5 per cent payable half-yearly for the remaining 5 years. Find the value of a half-yearly payment?

24. A man pays 50 cents at the beginning of each week as a premium on an industrial insurance policy. If money is worth 4 per cent and interest can be compounded weekly, what would be the accumulated amount of the premiums at the end of a year if the man lives to pay all the 52 premiums?

25. What single payment at the beginning of the year is equivalent to the weekly payments of 50 cents for a year as in problem 24 if the man lives to pay all of the 52 premiums?

CHAPTER III

THE SINKING FUND METHOD OF PAYING A DEBT BY PERIODICAL INSTALMENTS

44. Meaning of a sinking fund. A **sinking fund** is a fund set aside for the purpose of meeting some future obligation, usually the payment of the principal of a debt when it falls due. There are two common methods of treating the problem of paying a debt by periodical payments. We may assume that the entire principal remains outstanding during the whole time, and that the sinking fund created by periodical payments is separately invested and is to accumulate to an amount that will suddenly extinguish the debt at the end of the time. The sum set aside for a sinking fund is often invested at a different rate of interest from that paid on the debt. For example, a man may borrow \$1,000 at 6 per cent for three years. If, after paying the interest of \$60, he deposits \$320.35 at the end of each year in a bank paying 4 per cent effective, he creates a sinking fund which will accumulate to \$1,000 in three years. If he could find a 6 per cent investment for the sinking fund he would deposit \$314.11 per annum besides paying the interest of \$60, a total payment of \$374.11 per annum. The payments \$320.35 and \$314.11 are simply the annual rents of annuities that will amount to \$1,000 in three years at 4 per cent and 6 per cent, respectively. In general, when the same amount is set aside at regular intervals of time, as is usually the case, the sums put into the sinking fund constitute an annuity certain to which the relations among present value, amount, term and rate of interest derived in Chapter II, apply.

Instead of leaving the entire principal of a debt standing until the end to be extinguished by a sinking fund, we may consider any payment over what is needed to pay interest on the

principal to be applied at once towards liquidation of the debt. As the debt is being paid off, a less and less amount is required for interest, so that with a uniform payment per year, a greater and greater amount is available to refund the principal. This method is called the method of **amortization of principal.** For example, in the case of $1,000 borrowed for three years at 6 per cent, the payment of $374.11 at the end of each year will pay interest and principal as detailed in the table, usually called an amortization schedule.

Year	Principal at Beginning of Year	Interest at 6 Per Cent	Annual Payment	Amount Paid on Principal
1	1,000.00	60.00	374.11	314.11
2	685.89	41.15	374.11	332.96
3	352.93	21.18	374.11	352.93
			Total	1,000.00

When the sinking fund bears the same rate of interest as the principal, the two methods differ only in form. There is a difference in bookkeeping but not in reality. The method of amortization is simply a special case of a sinking fund invested with the owner of the security at the same rate of interest as that paid on the principal.

45. Annual payments. The problem of finding the annual payment into a sinking fund to pay a debt P due in n years may be discussed in three cases.

1. *When the debt P bears no interest.* This problem is simply that of finding the annual rent R of an annuity which in n years at the effective rate i will amount to P, that is, the problem of Art. 41.

For this case, $$R = P \cdot \frac{1}{s_{\overline{n}|}}. \tag{1}$$

EXAMPLE. What annual deposit must be made in a savings bank paying 4 per cent effective to accumulate \$1,000 in the three years?

SOLUTION:—

$$R = \$1{,}000 \quad \frac{1}{s_{\overline{3}|}} \text{ at 4 per cent.}$$

From Table V, we find

$$\frac{1}{s_{\overline{3}|}} = \frac{1}{a_{\overline{3}|}} - .04 = 0.3203485.$$

Hence, $$R = \$320.35.$$

2. *When the debt P bears the same rate of interest as the sinking fund.* Here the annual payment will be the annual rent of an annuity that will amount to P in n years plus the interest per year on P, both with interest rate i. Thus,

$$R = P \cdot \frac{1}{s_{\overline{n}|}} + iP = P\left(\frac{1}{s_{\overline{n}|}} + i\right) = P \frac{1}{a_{\overline{n}|}}. \tag{2}$$

The last result could be derived directly, for evidently the present value of the n annual payments should equal P. That is,

$$Ra_{\overline{n}|} = P.$$

EXAMPLE. What annual payment will extinguish a debt of \$1,000 principal and interest in 3 years at 6 per cent for both principal and sinking fund?

SOLUTION:—

$$R = 1000 \cdot \frac{1}{a_{\overline{3}|}} \text{ at 6 per cent.}$$

From Table V, $$\frac{1}{a_{\overline{3}|}} = 0.3741098.$$

Hence, $$R = \$374.11.$$

3. *When the rate of interest on the debt P is not the same as the rate by which the sinking fund is accumulated.* Let the rate of interest for P be i, and that for the sinking fund i'. The annual payment will be the annual rent of an annuity that will amount

to P in n years at rate i' plus the interest for a year on P at rate i. Thus,

$$R = P \cdot \frac{1}{s_{\overline{n}|}} \text{ (at rate } i') + iP.$$

Since $\dfrac{1}{s_{\overline{n}|}} = \dfrac{1}{a_{\overline{n}|}} - i'$, we have

$$R = P(i - i') + P\left(\frac{1}{a_{\overline{n}|}} \text{ at rate } i'\right). \tag{3}$$

Example. What annual payment will extinguish a debt of \$1,000 principal and interest in 3 years, if the principal bears interest at 6 per cent and the sinking fund can be accumulated at 4 per cent?

Solution:

$$R = 1000(.06 - .04) + 1000 \quad \frac{1}{a_{\overline{3}|}} \text{ at 4 per cent,}$$

$$= 20 + 1000 \cdot 0.3603485,$$

$$= \$380.35.$$

PROBLEMS

1. What annual payment must be made to accumulate \$1,000 in 3 years if the fund can be invested at 6 per cent effective? *Ans.* \$314.11.

2. What semiannual payment must be made to accumulate \$1,000 in 3 years, if the fund can be invested at 6 per cent nominal convertible semiannually? *Ans.* \$154.60.

3. A man buys a house giving a mortgage for \$5,000 at 6 per cent. He wishes to pay this off in 6 years. If he deposits money in a savings bank paying 3 per cent effective, what sum must be set aside annually to provide for payment of interest and extinction of the debt in 6 years? *Ans.* \$1,072.99.

4. A man wishes to pay a debt of \$1,250, principal and interest, in 8 equal annual instalments, the first payment being made immediately. If the rate of interest is 5 per cent what is the annual payment? *Ans.* \$184.19.

5. Construct an amortization schedule for the payment of a debt of \$3,214.65 in 5 years if money is worth 5 per cent effective.

6. Derive an expression for the payment that must be set aside every pth part of a year to extinguish a debt P in n years if P bears no interest and money set aside is at an effective rate i.

7. Derive an expression for the payment that must be set aside every pth part of a year to pay principal and interest on a debt P in n years. Let the interest on the principal be at the same rate j convertible p times a year as that used to accumulate the sinking fund.

46. Amount remaining due after the *m*th payment is made. In many business institutions dealing with loans paid by the amortization process it is necessary at times to know the exact amount of indebtedness after a certain number of payments have been made. It is also often desirable to close up a transaction involving amortization of the principal. In either case it is necessary to know the part of the principal unpaid. In general terms the problem is that of finding the amount remaining due after the mth payment has been made.

Let A_m represent that part of the principal remaining unpaid after the mth payment has been made. The $(n-m)$ payments still due form an annuity, and the debt could be cancelled by paying the present value of this annuity. This present value will then be exactly the amount due after the mth payment has been made. If R is the annual payment, the present value is $Ra_{\overline{n-m|}}$, and

$$A_m = Ra_{\overline{n-m|}}. \tag{4}$$

47. Amortization schedules. In the amortization of loans, it is convenient for some purposes to prepare schedules showing for any given year just what part of the annual payment is for repayment of principal and what part is for interest as well as showing the part A_m of the principal remaining due just after the annual payment has been made.

If the principal at the beginning is P, and is to be paid by n equal annual payments of R each, we have

$$R = \frac{P}{a_{\overline{n|}}}.$$

The schedule could now easily be exhibited in general terms by accumulating the outstanding principal at interest for successive years and subtracting the payments. But the processes involved in schedule making seem to be fixed more clearly in

the mind of the student through less formal treatment by means of numerical examples.

EXAMPLE. Construct a schedule for the amortization of a debt of \$2,500 payable annually in 7 annual instalments with interest at 6 per cent.

SOLUTION: From the above equation

$$R = 2500 \cdot \frac{1}{a_{\overline{n}|}} \text{ at 6 per cent,}$$

$$= (2500)(0.1791350),$$

$$= \$447.837.$$

The interest for the year is \$150.00 leaving

$$\$447.84 - \$150.00 = \$297.84$$

for payment on the principal. The outstanding principal at the beginning of the second year is \$2,500 — \$297.84 = \$2,202.16. Starting with this principal, we repeat the process for the second line of the schedule and so on for the remainder of the table. Theoretically the column headed "paid on principal" should total to the original principal. However, if as in the accompanying table, the different items are carried out only to cents, we may expect a slight discrepancy. The original principal added to the total interest should equal the total payments, *i. e.*, the total of the annual payment column. The whole amount of interest paid should equal the interest on the total of the "principal at beginning of year" column. These statements may be used as checks on the accuracy of the schedule.

Schedule showing the progress in the payment of a debt of \$2,500.00 principal and interest in 7 annual instalments with interest at 6 per cent.

Year	Principal at Beginning of Year	Annual Payment R	The Parts of the Annual Payment	
			Interest for Year	Paid on Principal
1	2,500.00	447.84	150.00	297.84
2	2,202.16	447.84	132.13	315.71
3	1,886.45	447.84	113.19	334.65
4	1,551.80	447.84	93.11	354.73
5	1,197.07	447.84	71.82	376.02
6	821.05	447.84	49.26	398.58
7	422.47	447.84	25.35	422.49
	\$10,581.00	\$3,134.88	\$634.86	\$2,500.02

48. Book Value. In keeping accounts connected with sinking fund operations, the outstanding principal at any time at which it may be calculated is often called the **book value** of the indebtedness at the date of entry on the books. In general, the word "book value" is somewhat broader in its meaning and includes the value at a given date of any item of the accounts which varies in value from entry to entry. For example, if a city borrows money to be repaid by some scheme of periodical payments, the city clerk must keep accounts showing the exact indebtedness of the city at least once a year. The books kept by the secretary of a building and loan association show, usually at half yearly intervals, the exact amount credited to each member. In listing assets of a company there are often items whose values change from year to year due to wear and tear. These entries are often listed under the heading "book value."

EXERCISES AND PROBLEMS

1. A borrower has been paying off a debt of $2,600 principal and interest in six equal annual payments with interest at 6 per cent. At the time of the fourth payment what amount is necessary to make the payment and to extinguish the entire debt? *Ans.* $1,498.14.

2. Prepare a schedule showing the principal and interest paid each year on the debt in problem 1, assuming the six payments had been made as originally planned.

3. A paving assessment of $845.19 is to be paid principal and interest in ten equal annual instalments. What is the book value of the indebtedness to the city at the time the sixth payment is due but not paid? Interest 5 per cent. *Ans.* $497.58.

4. A man borrows $5,000 at 6 per cent effective. At the end of the first year he pays $1,000, at the end of the second $2,000, at the end of the third and fourth years $1,000. What payment should be made at the end of the fifth year to extinguish the debt.

5. Prepare a schedule showing the principal and interest paid each year on the debt in problem 4.

6. A machine costing $1,200, supposed to wear out in 15 years is to be replaced by the accumulation at 4 per cent of a sinking fund made up of annual payments during the life of the machine. At the end of the eighth year just after the eighth payment has been made, what is the book value of the machine?

7. A newly organized factory borrows $20,000 at 6 per cent to be repaid principal and interest in 10 equal annual instalments the first payment to be made at the end of the second year. Prepare a schedule showing the yearly payment for principal and interest.

49. Amount in the sinking fund at any time. To find the amount in the sinking fund at any time, we have simply to find the amount of an annuity.

Let S_m represent the amount in the sinking fund at the end of m years, P the principal, n the time during which the sinking fund is to accumulate, and i the rate of interest. If the debt P bears no interest, the annual payment is [(1) Art. 45],

$$\frac{P}{s_{\overline{n}|}}.$$

Hence,
$$S_m = P\frac{s_{\overline{m}|}}{s_{\overline{n}|}}, \tag{5}$$

or
$$S_m = P\frac{(1+i)^m - 1}{(1+i)^n - 1}.$$

If P bears the same rate of interest as the sinking fund, the annual payment including interest on P is [(2), Art. 45]

$$\frac{P}{a_{\overline{n}|}}.$$

Hence,
$$S_m = P\frac{s_{\overline{m}|}}{a_{\overline{n}|}}, \tag{6}$$

if the interest has been allowed to accumulate and is to be paid from the sinking fund. If the interest had been paid annually, so that no interest is to be paid out of the sinking fund, the amount S_m in the sinking fund would be the same as in the first case.

If P bears interest at rate i and the sinking fund is accumulated at rate i', the annual payment, including the interest on P, is

$$\frac{P}{s_{\overline{n}|}} \text{ at rate } i' + iP.$$

Hence,

$$S_m = P\frac{s_{\overline{m}|}}{s_{\overline{n}|}} \text{ at rate } i' + i\,P\,s_{\overline{m}|} \text{ at rate } i', \tag{7}$$

if interest has been allowed to accumulate, and is to be paid from the sinking fund. If interest had been paid annually, so that no interest is to be paid from the sinking fund, the amount in the sinking fund would be the same as in the first case.

Exercise. A debt of \$2000 bearing interest at 6 per cent is due in eight years. A sinking fund bearing interest at 5 per cent, is created by equal annual payments to discharge this debt, principal and interest. Find the amount in the sinking fund just after the fifth annual payment into fund.

Form for Solution: In this case,

$$m = 5,\ n = 8,\ i = .06,\ i' = .05,\ P = 2000.$$

Hence,

$$S_5 = \$2000\frac{s_{\overline{5}|}}{s_{\overline{8}|}} \text{ at rate } .05 + .06(2000)\ s_{\overline{5}|} \text{ at rate } .05,$$
$$= ?$$

50. Retirement of bonded debt when bonds sell at a premium or a discount. If a borrower instead of paying money into a sinking fund wishes to use the money to reduce the principal, but there is no provision for making payments before the principal is due, he must obtain the consent of the lender. If the lender had expected the interest on the principal for the entire time he would probably not consent to receive the part payment without some consideration usually called a premium. The case may be reversed, and the lender, needing the money, may allow a discount from the face value of the portion of the debt paid. In either case we may have the problem of finding the equal annual payments required to pay interest and to retire the debt at the required date at the same time allowing for premium or discount.

The problem can be more easily stated if we consider the debt as a bonded debt, that is in the form of notes of convenient denominations called bonds all due at the same time. If these bonds are selling on the market for an amount different from the face value they are said to sell at a premium or at a dis-

count according as the price is greater than or less than face value. Our problem is then to determine the equal annual payments necessary to pay interest on a bonded debt and to buy back at the market price a sufficient number of bonds so that all will be retired at the time the debt is due. It will be assumed that the market price of the bonds is constant during the whole period.

Let P be the principal, i the rate of interest, R the annual payment and b the market price per unit of the face value of the bonds.

If R' is the annual payment when the securities were purchased at face value, then

$$R' = \frac{P}{a_{\overline{n}|} \text{ at rate } i},$$

in which P may be considered as the present value of the future payments. When they are purchased at b per unit of face value, the rate of interest changes from i to $\frac{i}{b}$, while the present value of the future payments is Pb. Hence,

$$Ra_{\overline{n}|} \text{ at rate } \frac{i}{b} = Pb;$$

that is,

$$R = \frac{Pb}{a_{\overline{n}|} \text{ at rate } \frac{i}{b}}. \tag{8}$$

Ordinarily $\frac{i}{b}$ is not likely to be a number for which tables are given, but the approximate value of $a_{\overline{n}|}$ $\left(\text{or } \frac{1}{a_{\overline{n}|}}\right)$ at rate $\frac{i}{b}$ can be found by interpolation by proportional parts when values of $a_{\overline{n}|}$ are given for rates between which $\frac{i}{b}$ is found. It is of course impossible to buy back bonds in fractional parts, so the equal yearly payments given by the above solution of this problem cannot be realized in practice. But in any actual transaction the result is a sort of guide in planning the repurchase of bonds

so that the total yearly payments will be as nearly equal as possible. The following example illustrates the application of the method.

EXAMPLE: A county borrows $50,000 for drainage purposes at 4 per cent by a bond issue in denominations of $1000 maturing in 10 years. The bonds sell on the market for $97 per $100 face value. Construct a schedule of payments of interest and repurchases of bonds so that the total amounts spent yearly will be as nearly equal as possible.

SOLUTION: Here $\frac{i}{b} = \frac{.04}{.97} = 0.041237.$

From Table V, $\frac{1}{a_{\overline{10}|}}$ at $.04 = 0.1232909,$

$\frac{1}{a_{\overline{10}|}}$ at $.0425 = 0.1248301.$

By interpolation, $\frac{1}{a_{\overline{10}|}}$ at $.041237 = 0.1240525.$

Hence, $R = (\$50000)(.97)(0.1240525) = \$6{,}016.55.$

The interest for the first year is $2000, leaving $4016.55 to buy bonds at $97. Four bonds only can be bought yearly in the first years, but as interest decreases more bonds can be purchased. The details are shown in the table.

Year	Principal Unpaid at Beginning of Year	Interest at 4 Per Cent	Number of Bonds Repurchased	Face Value of Bonds Retired	Cost of Bonds	Annual Payment
1	$50,000	$2000	4	$4000	$3880	$5880
2	46,000	1840	4	4000	3880	5720
3	42,000	1680	4	4000	3880	5560
4	38,000	1520	5	5000	4850	6370
5	33,000	1320	5	5000	4850	6170
6	28,000	1120	5	5000	4850	5970
7	23,000	920	5	5000	4850	5770
8	18,000	720	6	6000	5820	6540
9	12,000	480	6	6000	5820	6300
10	6000	240	6	6000	5820	6060
Totals	$296,000	$11,840	50	$50,000	$48,500	$60,340

The totals may be used to check the arithmetic. For example, the interest at 4 per cent on the total of the second column should equal the total of the third column. If the price of the bonds changes, a corresponding change must be made.

In case a more accurate value of R is desired or a check is required on the interpolation, substitute for $a_{\overline{n}|}$ in (8), its value

$$a_{\overline{n}|} = \frac{1 - \left(1 + \dfrac{i}{b}\right)^{-n}}{\dfrac{i}{b}} = \frac{b}{i}\,\frac{(b+i)^n - b^n}{(b+i)^n},$$

obtaining
$$R = i\,P\,\frac{(b+i)^n}{(b+i)^n - b^n}. \tag{9}$$

For the above example, we find

$$\begin{aligned} R &= .04\cdot 50000\left[\frac{(1.01)^{10}}{(1.01)^{10} - (.97)^{10}}\right] \\ &= \$6{,}016.49, \end{aligned}$$

but this value of R will lead to the same schedule as the value obtained by interpolation.

PROBLEMS

1. In order to save \$1000 in ten years a man deposits at the end of each year a certain sum in a savings bank paying 3 per cent effective. At the beginning of the eighth year what amount has he on deposit? *Ans.* \$668.40.

2. In order to save \$1000 in ten years, a man deposits at the end of each half year a certain sum in a savings bank paying 3 per cent nominal convertible semiannually. At the beginning of the eighth year what amount has he on deposit? *Ans.* \$668.23.

3. A man borrows \$1,500 at 6 per cent interest to be paid principal and interest in eight equal annual instalments. The lender invests the payments as he receives them in a 6 per cent investment. To what do the payments amount at the end of five years? *Ans.* \$1,361.64.

4. A debt of \$2,500 with accumulated interest at 6 per cent is to be paid by the accumulation of seven equal annual payments into a sinking fund bearing interest at 4 per cent. If the payments are deposited in a savings bank paying 4 per cent effective, to what do they accumulate in four years?

5. Reconstruct the schedule of the example of Art. 50 if the bonds had been issued in denominations of \$500.

6. Reconstruct the schedule of the example of Art. 50 if the bonds had been issued in denominations of \$100.

7. A city issues 5 per cent paving bonds for \$100,000 in denominations of \$1,000 maturing in 15 years. The bonds sell on the market at 101 for eight years, when they drop to 99. Construct a schedule of payments of interest and repurchases of the bonds so that the total amounts spent yearly will be as nearly equal as possible.

MISCELLANEOUS PROBLEMS

1. In order to save money to buy an automobile costing \$2,765, a man deposits at the end of every six months a certain sum in a savings bank paying 3 per cent nominal convertible semiannually. He wishes to buy the automobile at the end of three years. What is the semiannual deposit? *Ans.* \$443.85.

2. If the man in problem 1 borrowed the money to buy the automobile now and arranged to pay the debt, principal and interest, in six equal semiannual payments beginning at the end of the first half year, what would be the payment if the interest rate was 6 per cent nominal convertible semiannnually? *Ans.* \$510.41.

3. If in problem 2, the man used the savings bank of problem 1 to accumulate a sinking fund, what sum must he have in hand at the end of each half year to pay interest and to extinguish the debt in three years? *Ans.* \$526.80.

4. A man borrows P dollars at rate i, and arranges to pay $2Pi$ dollars per year until the debt is paid. Construct a schedule showing the principal and interest paid each year for the first four years.

5. A man borrows \$1000 at 5 per cent and arranges to pay twice the amount of the interest on the original principal each year until the debt is extinguished. Construct a schedule for the repayment of the loan.

6. A man buys a house for \$12,000 paying one-half down. He arranges to pay \$1500 per year principal and interest on the remaining amount at 6 per cent until the debt is paid. How many payments of \$1500 are made and what was the payment at the end of the year of settlement?

7. At the end of two years what was the owner's equity in the house of problem 6?

8. A county borrows \$25,000 to build a bridge. The payment is to be made by amortization of the principal for 15 years at 6 per cent. At the end of the tenth year what part of the debt is unpaid?

9. A sidewalk tax of \$174.83 is to be paid in five equal annual instalments with interest at 5 per cent. Construct a schedule of the payments, showing each year the interest payment, book value of the indebtedness and book value of credit on the tax to the taxpayer.

10. A city borrows \$20,000 to build a market. The debt is in the form of bonds of face value \$500 bearing interest at 5 per cent effective for 6 years. The bonds sell on the market at \$102 per \$100 of face value. What amount must be set aside annually to pay interest and purchase bonds so that the debt will be extinguished in 6 years? Since the bonds cannot be

bought in any amount except multiples of \$500, construct a schedule so that the total annual payments shall be as nearly equal as possible.

11. An automobile truck costing \$2000 and lasting five years at the end of which time it has a secondhand value of \$350 is to be replaced by means of a sinking fund accumulated at 4 per cent from annual payments during the 5 years. At the beginning of the fourth year the truck is destroyed by accident with a junk value of \$50. What amount of money must be added to the sinking fund and junk value to purchase a new truck for \$2000? *Ans.* \$999.07.

12. A fraternity chapter borrows \$30,000 at 7 per cent to build a house. A sinking fund can be built up at 5 per cent. What amount must be raised yearly to pay this debt if arrangements are made to extend the payments over 30 years?

13. Champaign county, Illinois, had in 1920 an assessed valuation of \$99,000,000. If \$1,000,000 were borrowed for road improvements at 5 per cent, to be paid by amortization of the principal in 25 years, how much will the county taxes be raised per dollar of valuation? *Ans.* 0.7 mills.

14. A debt P is to be paid principal and interest in n years by equal payments made p times a year. Let i be the effective rate of interest. Show that the periodical payment is

$$\frac{P}{pi} \cdot j_{(p)} \quad \frac{1}{a_{\overline{n}|}}.$$

15. A man borrows \$1000 at 5 per cent and agrees to pay one-fifth of the original principal for interest and on principal each year until the debt is paid. How long will it take to pay the debt?

16. Derive a formula for the time required to pay a non-interest bearing debt P by means of annual payments, R, allowed to accumulate at effective rate i.

17. Derive a formula for the time required to pay an interest-bearing debt P by means of annual payments, R, allowed to accumulate at the same rate as the interest on P.

18. A man pays \$1000 per year for four years, and then \$2000 per year for four years on a debt of \$10,000 bearing interest at 6 per cent. What part of the debt is unpaid at the end of the eight years? *Ans.* \$1666.40.

19. According to the estimates of the U. S. Treasury, it will cost the American people about \$1,200,000,000 a year for the next twenty-five years to pay off the war debt. This calculation is based on the assumption that the net war debt with deductions of loans to the allies will be eighteen billion dollars, and that the interest will be 4¼ per cent. This result would require besides interest payments an annual payment of 2.32 per cent of the principal. Check these figures.

CHAPTER IV

VALUATION OF BONDS AND OTHER SECURITIES

51. Introduction. For the present purpose, a **bond** may be defined as a promise to pay a definite sum due on a given date, and to pay interest or dividends in the meantime on the sum at a given rate, and at specified intervals.

The interval between interest payments is most commonly a half year, but it may be a year or some other time.

The sum named in the face of the bond is sometimes called its **face value** but more commonly its **par value.** When the sum due is repaid as specified in the bond, the bond is surrendered to the debtor and it is said to be **redeemed.**

If the market price of a bond is the same as its face value, it is said to be **at par**; if more than its face value, it is said to be **at a premium**; if less than its face value, it is said to be **at a discount.** Bonds are redeemed at par when nothing is specified to the contrary. In some cases it is specified that they are redeemed at a figure above par. This is usually a device to make the bonds appear more attractive to the prospective purchaser.

52. The investment rate of interest determined by certain factors. Bonds are usually bought as an investment to yield the purchaser a rate of interest known as the **investment rate** or income rate. The investment rate may be very different from the rate of interest named in the bond. To avoid confusion, it has been suggested that where misunderstanding is likely, it would be well to say **rate of dividend** instead of rate of interest for the rate named in the bond. We shall follow this suggestion.

The investment rate depends upon various considerations of which the following are important:

(a) The extent and nature of the security offered for the prompt payment of the debt at the due date.

(b) The ability of the issuing government, municipality, corporation, or individual to pay the interest as it falls due.

(c) The price at which the bond is to be redeemed.

(d) The interval of time before the bond is to be redeemed.

(e) The marketability of the bond if the purchaser wishes to sell it at any time.

(f) The current and probable future interest rate.

(g) The rate of interest or dividend named in the bond.

After considering these factors and any others that may be involved, the purchaser should decide on his investment rate for a security and be willing to pay a price that gives that rate. The determination of the price to yield a given rate is one of the important applications of compound interest and annuities.

53. The price of a bond to yield a given rate of interest.

Let C = the price at which the bond is to be redeemed,

n = the number of years to redemption date,

g = the ratio of total interest or dividends paid per year to the redemption price C,

i = the effective investment rate.

From the standpoint of the purchaser, the value of a bond consists of two parts:

(1) The interest to be received.

(2) The sum to be redeemed.

When the interest on the bond is payable p times a year, the interest consists of $\frac{Cg}{p}$ payable p times each year for n years. These interest payments are thus an annuity of annual rent Cg payable in p equal instalments per year.

Hence, the present value of the interest is

$$Cga_{\overline{n}|}^{(p)},$$

where $a_{\overline{n}|}^{(p)}$ is calculated at an effective rate i.

Besides these interest payments, we have for the present value of the redemption price

$$Cv^n,$$

where
$$v = \frac{1}{1+i}.$$

Hence, when we add together the present values of the redemption price and interest payments, we have for the purchase price of the bond, to yield an effective rate i,

$$A = Cv^n + gCa_{\overline{n}|}^{(p)} \text{ at rate } i. \tag{1}$$

On account of the usual practice with respect to intervals between payments of interest on bonds, and the form of quotation of the investment rate of interest, it seems best to discuss (1) under three cases:

CASE I. *When the interest on the bond is payable annually.* In this case, $p = 1$ in (1) and we have

$$A = Cv^n + gCa_{\overline{n}|}.$$

Substituting in this relation

$$a_{\overline{n}|} = \frac{1 - v^n}{i}, \text{ we have}$$

$$A = Cv^n + \frac{g}{i}(C - Cv^n).$$

If we make $Cv^n = K$, we have

$$A = K + \frac{g}{i}(C - K), \tag{2}$$

where K is the present value of the redemption price.

This formula was first established by Makeham—a well-known English actuary.

In addition to the algebraic derivation just given, it may be established by the following reasoning: The value of the bond is the sum of the present value K of the redemption price, and the present value of the interest payments. If the interest

rate named in the bond were i instead of g, the purchase price would be simply C, and $C - K$ would be the present value of the interest payments. Obviously, the present value of interest payments at rate g bears the same ratio to $C - K$ that g bears to i. Hence, the present value of the interest payments is $\frac{g}{i}(C - K)$, and the two parts together give $K + \frac{g}{i}(C - K)$.

The excess* at which the purchase is made is given by subtracting C from each member of (2). This gives

$$A - C = \frac{g - i}{i}(C - K). \tag{3}$$

By making $C = 1$ in (3), we have the purchase price per unit paid at redemption. This gives

$$A = 1 + \frac{g - i}{i}(1 - v^n) = 1 + (g - i)\, a_{\overline{n}|}, \tag{4}$$

where $a_{\overline{n}|}$ is to be taken at rate i. If the excess per unit of redemption price is denoted by k, we have

$$k = (g - i)\, a_{\overline{n}|}, \tag{5}$$

where $a_{\overline{n}|}$ is to be taken at rate i. From this relation it is clear that k is positive when $g > i$, and it is negative when $g < i$. If the bond is redeemed at par, k is a premium when $g > i$, and it is a discount when $g < i$.

It should be observed from (5) that the excess per unit of redemption price is simply the present value of an annuity of annual rent $g - i$ for the time n. Furthermore, annual instalments of $g - i$ can be set aside by the purchaser to **write off** or **amortize** the excess he paid.

*We use the term "excess" to denote excess of purchase price over redemption price. This excess has sometimes been called the "premium" or the "premium on the redemption price." See *Text-Book*, Institute of Actuaries, Part I, p. 73, Edition 1915. We use the term "excess" to avoid the confusion of using the term "premium" in two different senses.

EXAMPLE: Find the purchase price per \$100 of face of a $4\frac{1}{4}$ per cent bond, interest payable annually, to be redeemed at par in 18 years, when the investment rate is 6 per cent effective.

SOLUTION: In this case, $n = 18$, $g = .0425$, $i = .06$, and by formula (5) the excess

$$k = (.0425 - .06)a_{\overline{18|}}^{6\%} = (-.0175)(10.8276),$$

$$= -.189483,$$

or a discount of nearly 19 cents on the dollar. For each \$1 of the bond, the purchaser pays

$$\$1 - \$.189483 = \$.810517.$$

The purchase price per \$100 is then \$81.05 to the nearest cent.

CASE II. *When the interest on the bond is payable m times a year and the investment rate is convertible m times a year.*

If the investment rate is quoted as a nominal rate j, and the dividends on the bond per unit of the sum to be redeemed are paid in m equal instalments per year, we may regard the n years as consisting of nm periods with $\frac{g}{m}$ as an interest payment per period and $\frac{j}{m}$ as an effective investment rate per period. Hence, in (5), we may substitute for g, i, and n the values $\frac{g}{m}$, $\frac{j}{m}$, and mn, respectively. This gives for the excess per unit to be redeemed

$$k = \left(\frac{g}{m} - \frac{j}{m}\right) a_{\overline{mn|}} \text{ at rate } \frac{j}{m}. \tag{6}$$

The case which occurs most frequently in practice is that in which the dividends and interest are on a semiannual basis, so that $m = 2$. This gives

$$k = \left(\frac{g}{2} - \frac{j}{2}\right) a_{\overline{2n|}} \text{ at rate } \frac{j}{2}. \tag{7}$$

EXAMPLE: Find the purchase price per \$100 of face of a $4\frac{1}{4}$ per cent bond, interest payable semiannually to be redeemed at par in 18 years, if it is purchased at an investment rate of 6 per cent per annum payable semiannually.

SOLUTION: In this case, $n = 18$, $m = 2$, $g = .0425$, $j = .06$, and by formula (7) the excess

$$k = (.02125 - .03)a_{\overline{36}|}^{3\%} = (-.00875)(21.83225) = -.191032.$$

For each dollar of the face, the purchase price is \$.808968.
Hence, the price per \$100 of face is \$80.90 to the nearest cent.

CASE III. *When the interest on the bond is payable p times a year and the investment rate is convertible m times a year, $m \neq p$.*

This case except for $m = 1$ does not occur sufficiently often in applications to give it much detailed consideration. However, the procedure would be to use the relation (1) by making the following substitutions in its right-hand member:

$$a_{\overline{n}|}^{(p)} = \frac{1 - v^n}{p\left[(1 + i)^{\frac{1}{p}} - 1\right]},$$

$$v = \frac{1}{1 + i},$$

and

$$1 + i = \left(1 + \frac{j}{m}\right)^m.$$

This gives

$$A = \frac{C}{\left(1 + \frac{j}{m}\right)^{mn}} + \frac{Cg}{p} \cdot \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1}. \tag{8}$$

In particular, when $m = 1$, we have $j = i$, and

$$A = \frac{C}{(1 + i)^n} + Cg \cdot \frac{i}{j_{(p)}} \cdot a_{\overline{n}|} \text{ at rate } i. \tag{9}$$

EXAMPLE: Find the purchase price per \$100 of face of a $4\frac{1}{4}$ per cent bond interest payable semiannually to be redeemed at par in 18 years, if it is to be purchased at an investment rate of 6 per cent per annum payable annually.

SOLUTION: In this case, $n = 18$, $g = .0425$, $j = .06$, $m = 1$, $p = 2$ and formula (8) gives

$$A = \frac{100}{(1.06)^{18}} + \frac{4.25}{2}\,\frac{1 - (1.06)^{-18}}{(1.06)^{\frac{1}{2}} - 1},$$

$$= \$35.034 + 4.25 \cdot \frac{.06}{j_{(2)}} \quad a_{\overline{18}|} \text{ at 6 per cent,}$$

$$= \$35.034 + (4.25)(1.01478)(10.8276) = \$81.73.$$

PROBLEMS

1. A 6 per cent \$5000 bond due in 20 years with semiannual coupons was bought to yield 5 per cent per annum payable semiannually. What is the premium per dollar if redeemed at par? What is the purchase price of the bond (1) if it is to be redeemed at par, (2) if it is to be redeemed at 105? *Ans.* 0.1255, \$5627.50, \$5720.61.

2. Find the purchase price of a \$10,000 bond for 20 years at 5 per cent interest payable annually so as to yield the investor an effective rate of 6 per cent. *Ans.* \$8853.01.

3. What is the purchase price of a \$10,000 bond bearing half yearly coupons of \$300 each and repayable in 25 years at a price of 110, when the investor is to realize 5 per cent convertible semiannually on his money. *Ans.* \$11,709.06.

4. Should an investor, who wished to make 5 per cent or more on his money, have bought Second Converted Liberty Loan bonds quoted at 85 on November 15, 1919? These bonds are redeemable November 15, 1942, and bear 4¼ per cent interest payable semiannually.

5. If a 3½ per cent government bond for \$1000 due in 23 years with half-yearly coupons is bought to yield 4 per cent * payable half-yearly, what is the purchase price of the bond? *Ans.* \$925.27.

6. A 4¼ per cent government bond due May 20, 1938, with half-yearly coupons is bought on May 20, 1920, to yield 6 per cent payable half-yearly. Find the discount and purchase price per \$100. What would be the purchase price on May 20, 1930, to make this bond yield the same rate?

7. Two bonds of \$1000 each due in 20 years are offered for sale at an investment rate of 6 per cent payable semiannually. The only difference between the two is that the one bears half-yearly interest coupons of \$25 and the other of \$35. What is the difference in purchase price?

8. By general reasoning, show from (4) Art. 53 that the difference between two bonds for the same amount due on the same date is simply the

* The expression "4 per cent payable half-yearly" is used as an abbreviation for "at the rate of 4 per cent per annum payable half-yearly."

present value of an annuity equal to the difference between their interest coupons.

9. Find the premium that must be paid July 15, 1920, on a bond of $100 bearing interest at 7 per cent nominal, payable January 15 and July 15, redeemable after 12 years at par, to yield 5 per cent convertible semi-annually.

54. Use of bond tables. Bond tables exist which give purchase prices for periods to maturity, for rates named in the bond, and for the investment rate. The times to maturity in some tables vary by half-year intervals from one to fifty years, and then by five-year intervals from fifty years to one hundred years. The bond rates are usually tabulated for a range from at least as low as 3 per cent to 6 or 7 per cent by steps of ½ per cent. The investment rates usually cover about the same range but by smaller steps, the usual interval being $^1/_{20}$ per cent. A brief portion of a table is shown here with steps of ¼ per cent between investment rates.

TABLE SHOWING PURCHASE PRICES TO NEAREST CENT, OF A 4 PER CENT BOND FOR $10,000 WITH INTEREST PAYABLE SEMIANNUALLY

Investment Rate Per Cent Compounded Semiannually	Time to Redemption Date			
	10 years	15 years	20 years	25 years
3.00	$10,858.43	$11,200.79	$11,495.79	$11,749.98
3.25	10,635.96	10,884.84	11,096.66	11,276.95
3.50	10,418.82	10,579.65	10,714.86	10,828.53
3.75	10,206.88	10,284.83	10,349.56	10,403.32
4.00	10,000.00	10,000.00	10,000.00	10,000.00
4.25	9,798.05	9,724.80	9,665.43	9,617.33
4.50	9,600.91	9,458.87	9,345.16	9,254.14
4.75	9,408.44	9,201.87	9,038.52	8,909.34
5.00	9,220.54	8,953.49	8,744.86	8,581.88

EXERCISES

1. Check by use of (7) Art. 53, the following purchase prices given in the table:

(*a*) Investment rate 5 per cent and time of redemption 20 years.

(*b*) Investment rate $3\frac{1}{2}$ per cent and time of redemption 20 years.

2. Consider a $5000 bond due in 20 years, and bearing half-yearly interest coupons at 4 per cent per annum. Find by the use of the above table the purchase price of such a bond on an interest basis of $4\frac{1}{2}$ per cent convertible half-yearly. Check the result by an independent calculation without the use of the table.

55. "Amortization of premium" at which a bond is bought. It is a fundamental principle of business that capital should not be impaired. Hence, when a bond is bought for more than the redemption value, provision should be made for restoring any excess of the original capital invested over the redemption value. The excess of interest on the bond over the interest required at the investment rate should be used for the gradual extinction of the excess book value over the redemption price. The value of a bond bought above redemption price thus diminishes at each interval until the date of redemption, at which time its book value is equal to the redemption price. This amortization of the excess of purchase price over redemption price is often called **amortization of the premium** or **writing off the premium,** although this expression conforms to our definition of premium only when the bond is redeemed at par.

To explain the amortization at the end of any interest payment interval, let B be the book value at the beginning of the interval, say a half-year interval, j the nominal rate of interest per annum, convertible half-yearly, to be realized by the purchaser, g the ratio of the dividends of the year to the redemption price C. The owner of the bond receives as dividend for the half-year $\frac{Cg}{2}$. Of this amount, $\frac{Bj}{2}$ is sufficient to give the expected interest, leaving

$$\frac{Cg}{2} - \frac{Bj}{2}$$

for the amortization of the premium at the end of the half-year.

To put the matter in more general form, for any interval, the amortization at the end of the interval is the amount of dividends or interest on bonds minus interest on book value at investment rate.

To illustrate, we may well apply this principle to some exercises in the preparation of schedules to show the amortization of premiums on particular bonds.

EXERCISES

1. Given a 7 per cent bond for $1000 issued Jan. 1, 1917, dividends payable semiannually, to be redeemed at par Jan. 1, 1922, and to yield the investor 5 per cent payable semiannually. Find the purchase price, and show by means of a schedule the amortization of the premium.

Solution: It is left for the student to show that the purchase price is $1087.52. The amortization at the end of the first half year is

$$(1000)(.035)-(1087.521)(.025) = \$7.81.$$

This amount should be written off the book value of the bond and bring it nearer to par. The book value is then $1087.52, — 7.81 = $1079.71. The amortization at the end of the next half-year is

$$(1000)(.035) - (1079.71)(.025) = \$8.01.$$

This amount is then written off the book value leaving $1079.71 — $8.01 = $1071.70.

This process should be continued in the preparation of the schedule shown below. If the various numbers in the schedule had been carried out to mills instead of cents the last item in the first column would have been $1000 instead of $999.99. (See schedule in exercise 1, Art. 56, where computations are carried out to mills.)

Schedule Showing Progress in Amortization of Premium

Date	Book Value or Principal Outstanding	Semiannual Interest = 2½% of Book Value	Semiannual Dividend = 3½% of Face Value	Amortization of Premium
1917 Jan. 1	$1087.52	$27.19	$35.00	$7.81
July 1	1079.71	26.99	35.00	8.01
1918 Jan. 1	1071.70	26.79	35.00	8.21
July 1	1063.49	26.59	35.00	8.41
1919 Jan. 1	1055.08	26.38	35.00	8.62
July 1	1046.46	26.16	35.00	8.84
1920 Jan. 1	1037.62	25.94	35.00	9.06
July 1	1028.56	25.71	35.00	9.29
1921 Jan. 1	1019.27	25.48	35.00	9.52
July 1	1009.75	25.24	35.00	9.76
1922 Jan. 1	999.99			

2. Given a 5½ per cent bond, interest payable half-yearly, to be redeemed in 10 years at 110. Find the purchase price of such a $1000 bond to yield the investor 4 per cent * payable semiannually, and show by means of a schedule the amortization of the excess price over the redemption value.

3. Construct a schedule of amortization for a 7 per cent bond with par value $100, payable in 6 years with interest dates Oct. 1 and April 1, purchased on the date of issue Oct. 1, 1920, so as to yield 6 per cent payable semiannually.

4. From funds belonging to a minor, the guardian buys at 103 nine $1000 five per cent bonds maturing in 7 years, interest payable semiannually. What annual sum can be used for the support of the minor assuming that the principal is to remain intact?

56. "Accumulation of discount" at which a bond is bought. If the dividend rate g based on the redemption value is less than the investment rate, the bond is purchased below redemption price. In this case, the bond rate is not sufficient to give the income demanded on money, so that the book value of the bond must be **written up** towards the redemption value as the date of redemption comes nearer. Writing up the book value to-

* The expression "4 per cent payable semiannually" is used to mean "at the rate of 4 per cent per annum payable semiannually."

wards the redemption value is often called the **accumulation of discount,** although this language conforms to our definition of discount only when the bond is redeemed at par.*

To explain the accumulation at the end of any period, we use B, j, C, and g with the meanings given in Art. 55. Then the accumulation or accrual at the end of such a half-year period is

$$\frac{Bj}{2} - \frac{Cg}{2}.$$

Similarly, for any period, the accumulation or accrual at the end of the period is found by subtracting the dividend on the bond from the interest on the book value at the investment rate.

EXERCISES

1. Given a 5 per cent bond for $1000 issued July 1, 1920, interest payable semiannually, to be redeemed July 1, 1926, and to yield the investor 6 per cent payable semiannually. Find the purchase price, and show by a schedule the increases in book value.

SOLUTION: The student should verify that the purchase price is $950.230. Carry each item in the table to mills.

Date	Book Value	Semiannual Interest = 3% of Book Value	Semiannual Dividend	Accumulation of Discount
1920 July 1	$950.230	$28.507	$25.000	$3.507
1921 Jan. 1	953.737	28.612	25.000	3.612
July 1	957.349	28.720	25.000	3.720
1922 Jan. 1	961.069	28.832	25.000	3.832
July 1	964.901	28.947	25.000	3.947
1923 Jan. 1	968.848	29.065	25.000	4.065
July 1	972.913	29.187	25.000	4.187
1924 Jan. 1	977.100	29.313	25.000	4.313
July 1	981.413	29.442	25.000	4.442
1925 Jan. 1	985.855	29.576	25.000	4.576
July 1	990.431	29.713	25.000	4.713
1926 Jan. 1	995.144	29.854	25.000	4.854
July 1	1000.00			

*See definition of discount, Art. 51. Some authorities have defined "discount" to mean a deviation below redemption value. See *Text-Book*, Institute of Actuaries, Part I, p. 74, Edition, 1915.

2. Given a $4\frac{3}{4}$ per cent Victory Loan Note for \$1000, dividends payable semiannually redeemable at par May 20, 1923. Find the purchase price May 20, 1920, to yield 6 per cent payable semiannually and prepare a schedule to show the accumulation of discount.

3. Given a 6 per cent bond for \$100, dividends payable semiannually, redeemable at 120 in 5 years. Find the purchase price so that it will yield 6 per cent payable semiannually, and prepare a schedule to show the progress of the book values.

4. On October 15, 1919, a church treasurer purchased at 93.10, six \$1000, $4\frac{1}{4}$ per cent Fourth Liberty Loan bonds from part of the endowment funds of the church, to yield 5 per cent. These bonds are redeemable on or before October 15, 1938. At this date the treasurer must show the original capital of \$5586 intact. Assuming that he will receive \$6000 Oct. 15, 1938, as the redemption price he can use for current expenses something more than the interest received from the bonds. Devise two plans giving the amount he is justified in taking each year from the other endowment funds for current expenses, assuming that all funds yield 5 per cent.

57. Bond purchased between interest payment dates. Thus far we have considered bonds bought either at the date of issue or at a date of interest payment. When bonds are purchased between interest payment dates, the seller is clearly entitled to a portion of the interest which would accrue between these dates. Two cases arise which require consideration.

CASE I. *When the bond is purchased at a certain price and accrued interest.* This statement is interpreted to mean accrued interest at the rate named in the bond, and is further interpreted to mean simple interest. The price is then that agreed upon plus the simple interest accrued from the last interest payment date to the date of purchase.

CASE II. *When the bond is purchased on a strictly yield basis.* This is interpreted to mean that the investment is to yield an assigned rate, say i, from the date of purchase.

Consider a bond for 1 bearing dividends g per period bought between dividend dates, say $\frac{p}{q}$ of a period after the dividend date to yield the purchaser i per period.

Let A be the purchase price, and A_0 the price just after the last dividend payment.

Then $$A = A_o (1+i)^{\frac{p}{q}} \tag{10}$$

gives the purchase price on a strictly yield basis. In practice simple interest is usually taken for the part of a period.

PROBLEMS

1. A bond of $1000 of the Seattle Lighting Company dated July 1, 1919, bearing 7 per cent interest payable semiannually was purchased March 1, 1920, at 98.5 and accrued interest. What was paid for the bond? *Ans.* $996.67.

2. The bond described in problem 1 is to mature July 1, 1929. What should have been paid for this bond March 1, 1920, if purchased to yield 7.2 per cent?

3. A bond of $10,000 issued Jan. 1, 1915, at 6 per cent payable semiannually and to be redeemed Jan. 1, 1925, was purchased Mar. 1, 1915, to yield 5 per cent. What should have been paid for the bond on that date?

4. What should have been paid for the bond in problem 3 if it had been sold at a price to yield 5 per cent as of date Jan. 1, 1915, plus accrued interest?

58. Determination of the investment rate from the purchase price of a given bond. We have considered in Arts. 53 and 54 the determination of the purchase price of a bond to yield the investor a given rate of interest. The converse problem—given the price to find the investment rate—is practically of great importance. To illustrate, a bond is usually quoted at a certain price without stating the corresponding investment rate. In order to determine the correct book value of the bond at the end of a given year, it is desirable to find first the investment rate.

When neither a bond table nor a table of annuities is available, the problem is likely to present some difficulty or at least to become somewhat laborious. Since bond tables * are now published for investment rates differing by $^1/_{20}$ of 1 per cent, over a considerable range of investment rates, and annuity tables † are pretty generally available for rates of interest differing by small amounts, it seems desirable to treat the problem first for the

* See Sprague, *Complete Bond Tables.*

† Spitzer, *Zinses-Zinsen-und Renten-Rechnung.*

situations in which all or part of the tables are available, and later for the situation in which no tables are available.

CASE I. *When a bond table is available.* Consider a 4 per cent bond for \$100 with dividends payable semiannually purchased at 90. If the bonds are to be redeemed at par in 15 years, what is the investment rate?

From the table in Art. 54, we note that such a bond would give an interest rate of

.0475 if sold at 92.0187 per 100 of face,
and .0500 if sold at 89.5349 per 100 of face.
Hence, by interpolation by proportional parts, a price of 90 corresponds to an investment rate of .0495 + .

For a closer approximation, from Sprague's Bond Tables with tabulations for investment rates that differ by only .005, we note that the bond would give an investment rate of

.0495 if sold for 90.02489 per 100 of face,
and .0500 if sold for 89.53485 per 100 of face.

Hence, by interpolation by proportional parts, a price of 90 corresponds to an investment rate of .049525. This is probably near enough for most purposes. This is the nominal rate $j_{(2)}$.

The effective rate per annum is given by

$$(1.024762)^2 - 1 = .0501372.$$

CASE II. *When no bond table is available but a table of annuities $s_{\overline{n|}}$ is available for rates of interest differing by small amounts.*

Consider the same bond as under Case I. With such a bond, there will be a gain of \$10 per \$100 at redemption because the bond was purchased at 90. To make the investment rate called for, we find the \$2 of semiannual dividends insufficient. To make good the difference between dividends and interest, at the investment rate, there must be available each half-year

$$\frac{\$10}{s_{\overline{30|}}} \text{ at } \frac{j_{(2)}}{2},$$

where $j_{(2)}$ is the nominal investment rate convertible twice a year.

Not knowing the investment rate, the plan is to try two rates

recorded in our table of annuities—one smaller than the investment rate and the other larger—but each as near to that rate as we are able to assign a value from the available table of annuities.

With a purchase price of 90 and 4 per cent dividends payable semiannually and redemption at par in 15 years, it is clear that the investment rate would exceed .045. Hence, we may well try .045 and .05 as investment rates.

Then we have

$$\frac{10}{s_{\overline{30}|} \text{ at } .0225} = 0.236993,$$

and

$$\frac{10}{s_{\overline{30}|} \text{ at } .025} = 0.227776.$$

But each dividend is worth \$2. At .045 assumed as an investment rate, the half-year return on 90 is 2.236993, or a half-yearly rate of $\frac{2.236993}{90} = .0248555$. Similarly, an assumed rate .05 leads to a half-yearly rate of .0247531. That is, an assumed rate .045 leads to a nominal yearly rate of $2(.0248555) = .0497110$ convertible half-yearly. Similarly, an assumed rate .05 leads to $2(.0247531) = .0495062$ convertible half-yearly.

Hence, by interpolation, the required rate $j_{(2)}$ is given by

$$\frac{j_{(2)} - .045}{.05 - .045} = \frac{j_{(2)} - .0497110}{.0495062 - .0497110},$$

or

$$\frac{j_{(2)} - .045}{.005} = \frac{.0497160 - j_{(2)}}{.0002048}.$$

Clearing of fractions, we have

$$.0052048\, j_{(2)} = .00025777.$$

$$j_{(2)} = .049524.$$

The result differs only slightly in the fifth significant figure from that obtained with bond tables.

CASE III. *When neither a bond table nor a table of annuities is available.* The premium on a bond of face C to be redeemed at par is, by Case II, Art. 53, given by

$$k_1 = C\left(\frac{g}{2} - \frac{j}{2}\right) a_{\overline{2n|}} \text{ at rate } \frac{j}{2}, \tag{11}$$

where the half-yearly rate of dividend is $\frac{g}{2}$ and the half-yearly investment rate is $\frac{j}{2}$, j being the nominal rate convertible half-yearly.

Since $a_{\overline{2n|}} = \dfrac{1 - \left(1 + \frac{j}{2}\right)^{-2n}}{\frac{j}{2}}$, we may by easy algebraic transformation, write (11) in the form

$$\frac{2k_1}{C(g - j)} = \frac{1 - \left(1 + \frac{j}{2}\right)^{-2n}}{\frac{j}{2}}. \tag{12}$$

Let k without a subscript be the premium for a unit of redemption price, so that $\frac{k_1}{C} = k$, then (12) becomes, if we invert each member,

$$\frac{g - j}{k} = \frac{j}{1 - \left(1 + \frac{j}{2}\right)^{-2n}}. \tag{13}$$

Expand $\left(1 + \frac{j}{2}\right)^{-2n}$ by the binomial theorem, and neglect the terms that are so small as to involve j^3 as a factor. Then (13) becomes

$$\frac{g - j}{k} = \frac{j}{nj - n(2n + 1)\frac{j^2}{4}}, \text{ approximately,} \tag{14}$$

$$= \frac{1}{n - n\,(2n+1)\frac{j}{4}}, \text{ approximately.}$$

Multiply each member by n, and we have

$$\frac{n\,(g-j)}{k} = \frac{1}{1 - \frac{2n+1}{4}j},$$

or
$$\frac{n\,(g-j)}{k} = 1 + \frac{2n+1}{4}j \text{ approximately.}$$

Solve for j and we get

$$j = \frac{4\,(ng - k)}{2n\,(k+2) + k}, \text{ approximately.} \tag{15}$$

Now, let us apply this result to the bond dealt with in Cases I and II. In this problem, $n = 15$, $g = .04$, $k = -.1$. With these values, we obtain from (15),

$$j = .0492.$$

This value serves well as a first approximation. Next, make $j = .0492 + h$, and follow a method similar to that of Art. 36 by the substitution of $j = .0492 + h$ in (13) with a view to finding h.

This gives

$$.092 + 10h = \frac{.0492 + h}{.51757 + 7.0629h},$$

by making $g = .04$, $k = -.1$, and retaining no terms in h of degree above one.

Clearing of fractions, and retaining no terms in h of degree above one, we have

$$4.826\,h = .0016,$$

$$h = .0003 +.$$

Hence, $j = .0492 + .0003 + = .0495 +$ gives a second approximation.

If a closer approximation is desired, make

$$j = .0495 + h,$$

and repeat the process by substitution in (13).

PROBLEMS

1. An insurance company purchased bonds bearing dividends at 6 per cent payable semiannually, due in 10 years at 109. What is the nominal investment rate, interest payable semiannually, if these bonds are redeemed at par? *Ans.* 4.853 + per cent.

2. In the situation described in problem 1, show by an amortization schedule, the book value of such a bond of face $1000 at the end of each half-year period up to the redemption date.

3. On July 1, 1920, an insurance company purchased a $10,000 bond bearing 6 per cent, payable semiannually due in 5 years, for $9000. What is the investment rate? Show by a schedule the accumulation of the discount.

59. Purchase price of bonds redeemed in instalments. When bonds are redeemed in instalments, it is obvious that the purchase price may be found by simply adding together the prices of the separate instalments. It is, in certain cases more convenient, however, to give a special treatment for such bonds.

CASE I. *When the dividends are payable annually at rate g on redemption price of outstanding bonds and the investment rate is an effective rate i per year.*

Let $C_1, C_2, \ldots\ldots, C_r$ be the successive instalments in which the bonds are to be redeemed;

$n_1, n_2, \ldots\ldots, n_r$ the corresponding number of years to the due dates of these instalments;

$K_1, K_2, \ldots\ldots, K_r$ the corresponding present values of the C's at investment rate i;

and $A_1, A_2, \ldots\ldots, A_r$ the corresponding purchase prices of the instalments as if they were separate bonds.

Then by (2), Art. 53, we have for the separate instalments,

$$A_1 = K_1 + \frac{g}{i}(C_1 - K_1),$$

$$A_2 = K_2 + \frac{g}{i}(C_2 - K_2),$$

$$\cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots$$

$$A_r = K_r + \frac{g}{i}(C_r - K_r).$$

Adding, we obtain

$$A_1 + A_2 + \ldots + A_r = K_1 + K_2 + \ldots + K_r$$
$$+ \frac{g}{i}[(C_1 + C_2 + \ldots + C_r) - (K_1 + K_2 + \ldots + K_r)].$$

This result may be written as follows

$$A = K + \frac{g}{i}(C - K), \tag{16}$$

where C is the total sum to be redeemed, K the total of the present values of $C_1, C_2, \ldots, C_r$, and A the total purchase price.

From (16), we obtain for the excess on a redemption price C,

$$A - C = (C - K)\frac{g - i}{i}. \tag{17}$$

For bonds redeemed at par, this excess is the premium.

If we wish the excess k for each unit of the redemption price we make $C = 1$ in (17) and obtain

$$k = (1 - K)\frac{g - i}{i}. \tag{18}$$

EXERCISE. Discuss the sign of k, (1) when g is greater than i, (2) when g is less than i. Give interpretation of a negative k.

CASE II. *When dividends are payable and interest is convertible m times a year.*

In this case we find it very simple to replace years in the development of formula (16) by periods between dividend payments. Then (16) still applies if the present value K is found at rate $\frac{j}{m}$ per period, and the dividend rate is taken at $\frac{g}{m}$ per period between payments of dividends.

EXERCISE 1. Show that formulas (16), (17), and (18) are not changed in form by replacing g by $\frac{g}{m}$ and i by $\frac{j}{m}$ at the same time.

EXERCISE 2. What should be bid for a \$1000 bond with dividends at 5 per cent payable semiannually maturing in two equal instalments at the ends of 10 and 15 years to yield 6 per cent payable semiannually?

Here $C = \$1000$, $K = \frac{500}{(1.03)^{20}} + \frac{500}{(1.03)^{30}}$, $g = .05$, $i = .06$.

60. Serial bonds. When bonds are redeemed in equal or nearly equal instalments, they are sometimes called **serial** bonds or the issues are called serial issues.

CASE I. *Bonds to be redeemed in r equal annual instalments beginning at the end of one year.*

We have in this case for a bond of redemption price 1, a part $\frac{1}{r}$ to be redeemed at the end of each year for r years. The present value of these instalments is

$$K = \frac{a_{\overline{r}|}}{r} \text{ at investment rate,}$$

and we have from (18) for the excess

$$k = \left(1 - \frac{a_{\overline{r}|}}{r}\right)\frac{g - i}{i}, \tag{19}$$

where $a_{\overline{r}|}$ is to be found at investment rate i. This formula is readily adapted to any case in which the intervals of interest payments, instalment redemptions, and interest conversions coincide. The year is simply replaced by the interval.

PROBLEMS

1. What is the purchase price of \$5000 of serial bonds issued Jan. 1, 1920, with 5 per cent coupons payable annually and maturing in 10 equal annual instalments, to net the purchaser an effective rate of 6 per cent?

SOLUTION: Here $r = 10$, $g = .05$, $i = .06$.
Then

$$k = \left(1 - \frac{a_{\overline{10}|}}{10}\right)\frac{.05 - .06}{.06},$$

$$a_{\overline{10}|} \text{ at rate } .06 = 7.3600870.$$

Hence, $k = -\frac{1}{6}(.2639913) = -.04399855,$

and the \$5000 should be purchased at a discount of

$$(5000)(.04399855) = \$219.993.$$

The purchase price $= 5000 - 219.993 = \$4780.007.$

2. Prepare a schedule to show the progress in the redemption, and in the accumulation of discount of bonds for which the purchase price has been found in problem 1.

TABLE SHOWING PROGRESS IN REDEMPTION OF BONDS AND IN ACCUMULATION OF DISCOUNT

Date	Book Value or Principal Outstanding	Interest for Year = .06 × Book Value	Dividend for Year = .05 × Face Value	Accumulation of Discount	Redemption Payments
1920, Jan. 1	\$4780.007	\$286.800	\$250.00	\$36.800	\$500
1921, Jan. 1	4316.807	259.008	225.00	34.008	500
1922, Jan. 1	3850.815	231.049	200.00	31.049	500
1923, Jan. 1	3381.864	202.912	175.00	27.912	500
1924, Jan. 1	2909.776	174.587	150.00	24.587	500
1925, Jan. 1	2434.363	146.062	125.00	21.062	500
1926, Jan. 1	1955.425	117.325	100.00	17.325	500
1927, Jan. 1	1472.750	88.365	75.00	13.365	500
1928, Jan. 1	986.115	59.167	50.00	9.167	500
1929, Jan. 1	495.282	29.717	25.00	4.717	500
1930, Jan. 1	000.00				

3. A loan of \$10,000 at 6 per cent payable semiannually is repayable in instalments of \$1250 each with each interest payment. Find the purchase price to yield the investor 5 per cent, payable semiannually.

4. Prepare a schedule showing progress of loan in problem 3.

CASE II. *Bonds to be redeemed in r equal instalments, the first redemption at the end of f years and the remaining redemptions at intervals of t years.*

Let g, the annual dividend for a redemption of 1, be payable in m equal instalments at intervals of $\frac{1}{m}$ year. We propose to derive from (18) a convenient formula for the price of these

bonds to yield the investor a nominal rate $j_{(m)}$. As in Case I, for a bond of redemption price 1, the present value of the instalments of $\frac{1}{r}$ each to be redeemed is

$$K = \frac{1}{r}\left[\frac{1}{\left(1+\frac{j}{m}\right)^{mf}} + \frac{1}{\left(1+\frac{j}{m}\right)^{mf+mt}} + \frac{1}{\left(1+\frac{j}{m}\right)^{mf+2mt}} + \cdots + \frac{1}{\left(1+\frac{j}{m}\right)^{mf+(r-1)mt}}\right] \tag{20}$$

The sum of the geometrical progression in the brackets is (See Art. 152)

$$\frac{\dfrac{1}{\left(1+\frac{j}{m}\right)^{mf}} - \dfrac{1}{\left(1+\frac{j}{m}\right)^{mf+mrt}}}{1 - \dfrac{1}{\left(1+\frac{j}{m}\right)^{mt}}}. \tag{21}$$

By adding and subtracting 1 in the numerator of (21), we have

$$\frac{1 - \dfrac{1}{\left(1+\frac{j}{m}\right)^{m(f+rt)}} - \left\{1 - \dfrac{1}{\left(1+\frac{j}{m}\right)^{mf}}\right\}}{1 - \dfrac{1}{\left(1+\frac{j}{m}\right)^{mt}}}$$

$$= \frac{a_{\overline{m(f+rt)}|} - a_{\overline{mf}|}}{a_{\overline{mt}|}},$$

where the annuities are at interest rate $\frac{j}{m}$. Hence,

$$K = \frac{a_{\overline{m(f+rt)}|} - a_{\overline{mf}|}}{r a_{\overline{mt}|}} \text{ at interest rate } \frac{j}{m}. \tag{22}$$

Consequently, from (18) and (22), the excess per 1 of redemption price is

$$k = \left[1 - \frac{a_{\overline{m(f+rt)}|} - a_{\overline{mf}|}}{ra_{\overline{mt}|}}\right]\frac{g - j}{j}, \tag{23}$$

where annuities are at rate $\frac{j}{m}$.

When $f = 1$, $t = 1$, and $m = 1$, we have Case I. Hence, Case I is simply a special case. However, it is clearly simpler to use (19) than (23) if $f = 1$, $t = 1$, and $m = 1$.

EXERCISES AND PROBLEMS

1. Write the special form taken by formula (23) for the very common case of $m = 2$.

2. Given \$100,000 highway serial bonds dated July 1, 1920, bearing 6 per cent coupons payable semiannually on outstanding face, to be redeemed in instalments of \$25,000 on July 1, 1923, July 1, 1925, July 1, 1927, July 1, 1929. What price should be paid for these bonds to yield 5 per cent, compounded semiannually? Prepare a schedule to show operation of funds in this loan.

SOLUTION: Here $f = 3$, $g = .06$, $m = 2$, $j_{(2)} = .05$, $r = 4$, $t = 2$.

By formula (23),

$$k = \left[1 - \frac{a_{\overline{22}|} - a_{\overline{6}|}}{4a_{\overline{4}|}}\right]\frac{.06 - .05}{.05},$$

where the annuities are at rate $\frac{j}{m} = .025$.

Substituting the values for the annuities from the tables, we have

$$k = 0.05038058.$$

Hence, the purchase price is \$105038.06.

Schedule Showing Progress of the Loan

Date	Book Value or Principal Outstanding	Semiannual Interest = 2½% of Book Value	Semi-annual Dividend = 3% of Face Value	For Amortization of Premium	Redemption Payments
1920, July 1	$105,038.06	$2625.95	$3000	$374.05	0.00
1921, Jan. 1	104,664.01	2616.60	3000	383.40	0.00
" July 1	104,280.61	2607.02	3000	392.98	0.00
1922, Jan. 1,	103,887.63	2597.19	3000	402.81	0.00
" July 1,	103,484.82	2587.12	3000	412.88	0.00
1923, Jan. 1,	103,071.94	2576.80	3000	423.20	$25,000.00
" July 1,	77,648.74	1941.22	2250	308.78	0.00
1924, Jan. 1,	77,339.96	1933.50	2250	316.50	0.00
" July 1,	77,023.46	1925.59	2250	324.41	0.00
1925, Jan. 1,	76,699.05	1917.48	2250	332.52	25,000.00
" July 1	51,366.53	1284.16	1500	215.84	0.00
1926, Jan. 1	51,150.69	1278.77	1500	221.23	0.00
" July 1,	50,929.46	1273.24	1500	226.76	0.00
1927, Jan. 1,	50,702.70	1267.57	1500	232.43	25,000.00
" July 1,	25,470.27	636.76	750	113.24	0.00
1928, Jan. 1	25,357.03	633.92	750	116.08	0.00
" July 1,	25,240.95	631.02	750	118.98	0.00
1929, Jan. 1	25,121.97	628.05	750	121.95	25,000.00
" July 1	00,000.02*				

3. Given a bond of $10,000 with 6 per cent coupons payable semiannually on outstanding face with the condition that $500 of the face is to be repaid with each payment of dividend, what is the purchase price to yield 7 per cent, convertible semiannually? Prepare a schedule to show the operation of the funds with respect to writing up discount and with respect to redemption. *Ans.* $9586.60.

4. Given a bond of $5000 with dividends at 6 per cent payable semiannually on outstanding face to run five years, and then to be redeemed by yearly instalments of $1000, the last instalment to be paid 10 years after the date of valuation. What is the purchase price to yield the investor 5 per cent, convertible semiannually? *Ans.* $5324.73.

* The balance of .02 could be eliminated by carrying one more figure in the calculations.

5. What would be the purchase price in problem 4 if the \$1000 instalments were redeemed at 120?

HINT: Note that $g = \dfrac{.06}{1.20} = .05.$

61. Annuity bonds. The expression "**annuity bonds**" is applied to bonds which are repaid at interest payment dates in such instalments that the sum of the interest and principal repaid is constant or as nearly constant as the denominations of the issue permit.

The problem of finding the instalment to be paid on such a bond is the same problem that is treated in Arts. 45, 47, where we have shown the operation of funds in the extinction of a debt by equal annual or semiannual payments at interest payment dates.

If C is the face value, n the time in years before the bond is entirely redeemed, and g the rate of interest named in the bond, each equal annual instalment of principal and interest together is, by Art. 45,

$$R = \frac{C}{a_{\overline{n}|}}, \tag{24}$$

where $a_{\overline{n}|}$ is at rate g.

The purchase price A of such a bond to be redeemed at par and to yield the investor a rate i is obviously

$$A = Ra_{\overline{n}|}, \tag{25}$$

where $a_{\overline{n}|}$ is at rate i.

If the bonds are to bear interest at rate g per annum, payable in m instalments, clearly each instalment is

$$R = \frac{C}{a_{\overline{mn}|}} \text{ at rate } \frac{g}{m}.$$

EXERCISES AND PROBLEMS

1. Find the purchase price of ten year annuity bonds for \$10,000 issued at rate 5 per cent per annum to yield 6 per cent. Prepare a schedule showing the progress in the repayment of the loan. *Ans.* \$9531.65.

2. Find the purchase price of a five year annuity bond for \$10,000 issued at nominal rate 5 per cent, payable semiannually, when bought to yield

4 per cent payable semiannually. Prepare a schedule to show the progress of the loan. *Ans.* \$10,263.39.

Since it is often desirable to subdivide a bond issue into individual bonds at \$50, \$100, \$200, \$500, or \$1000, it follows that the principal in annuity bonds is retired in multiples of the denomination of individual bonds. It is then necessary to adjust the schedule of redemption to meet this requirement. The method of making such adjustments should be clear from exercises.

3. Given that bonds such as are described in problem 2 are in denominations of \$100; prepare a schedule showing the adjustments to multiples of \$100 in the retirement of bonds at the end of each half-year so as to make the sum of each dividend and the corresponding repayment of principal nearly equal. From the result of problem 2, we obtain for each equal instalment of interest plus principal

$$R = \$1142.59.$$

Since, when the entire principal is outstanding, the semiannual dividend is \$250, it seems that we should retire about \$900 per half-year in the early part of the period and in the latter part of the period about \$1100.

Let us choose to retire \$900 for each of the first three half-years, then \$1000 for each of four half-years, and then \$1100 for each of three half-years.

The purchase price clearly depends to some extent on the particular adjustments made.

We should therefore calculate the purchase price to correspond to the plan of retiring bonds which we may have adopted. We may apply formula (23) to the three bonds of \$900 each, then to the four bonds of \$1000 each, and lastly to the three bonds of \$1100 each.

We thus find the following premiums:

Premium for the three of \$900 each	=	26.13
Premium for the four of \$1000 each	=	102.97
Premium for the three of \$1100 each	=	134.59
Total premium	=	\$263.69.

SCHEDULE ADJUSTED TO BONDS OF DENOMINATION $100

Year	Book Value or Principal at Beginning of Half-Year	Semiannual Interest of .02 × Book Value	Annuity Instalments	Amortization of Premium	Face of Bonds Retired at end of Half-Year
½	$10263.69	$205.27	$1150.00	$44.73	$ 900
1	9318.96	186.38	1127.50	41.12	900
1½	8377.84	167.56	1105.00	37.44	900
2	7440.40	148.81	1082.50	33.69	1000
2½	6406.71	128.13	1157.50	29.37	1000
3	5377.34	107.55	1132.50	24.95	1000
3½	4352.39	87.05	1107.50	20.45	1000
4	3331.94	66.64	1082.50	15.86	1100
4½	2216.08	44.32	1155.00	10.68	1100
5	1105.40	22.11	1127.50	5.39	1100

4. Prepare a schedule showing the progress of a loan in the form of an annuity bond issue of $100,000, denomination $100, bearing 7 per cent interest, payable semiannually, and retired in four years by eight semiannual annuity instalments as nearly equal as possible, when the bonds are purchased to yield 6 per cent payable semiannually.

MISCELLANEOUS PROBLEMS

1. The Victory Liberty Notes bear interest at 4¾ per cent payable semiannually. What should be paid for such a note of $500 three years before it is to be redeemed to yield the investor 6 per cent convertible semiannually? 6 per cent effective per annum? *Ans.* $483.07, $484.23.

2. If a Victory Liberty Note could be purchased at 96 three years before it is to be redeemed, what effective rate of interest could be realized by a purchaser of such a note? *Ans.* 6.17 per cent.

3. Prepare a schedule to show the operation of funds invested in the note described in problem 1.

4. A corporation can raise money by issuing a twenty year 6 per cent bond to be sold at 98. For the redemption, it must provide a sinking fund which will accumulate at 4 per cent. The corporation could raise the same sum by issuing twenty year 7 per cent annuity bonds to be sold at par. All interest rates are convertible semiannually. Which is the better financial plan for the corporation? What is the present value at 7 per cent of the difference in raising $98,000? *Ans.* $1,420.11.

5. A corporation was planning to raise \$100,000 by issuing bonds July 1, 1920, at 7 per cent payable half-yearly and to provide for the redemption, July 1, 1925, by placing money each half-year in a sinking fund at 5 per cent payable half-yearly. This corporation could also raise the money by selling 5 year 6 per cent annuity bonds at par. What is the value of the financial advantage of the one plan over the other at each interest paying date?

6. Ten years ago two bonds of \$10,000 each were issued at 5 per cent payable semiannually, the one an annuity bond for twenty years, and the other a serial bond with repayment of \$250 at each interest paying date. A party has agreed to purchase these bonds so as to yield 7 per cent payable semiannually. What should he pay for each bond just after this twentieth interest paying date?

7. An insurance company bought a \$10,000 bond paying dividends at the rate of 7 per cent payable semiannually at a price to yield the company at the rate of 6 per cent payable semiannually. If the bond is redeemed at par in 4 years, find the purchase price, and show by an amortization schedule the progress of the funds until the date of redemption.

8. A Victory Liberty Loan Note for \$500 is purchased Sept. 15, 1920, to yield an investor 5½ per cent convertible semiannually. It bears dividend coupons as follows:

Dec. 15, 1920......	\$11.88,
June 15, 1921......	11.87,
Dec. 15, 1921......	11.88,
June 15, 1922......	11.87,
Dec. 15, 1922......	11.88,
May 20, 1923......	10.18.

The note must be redeemed May 20, 1923, at par. Find the purchase price Sept. 15, 1920.

9. The Fourth Liberty Loan bonds bear 4¼ per cent coupons payable semiannually. These bonds may be redeemed at the option of the United States Government on and after Oct. 15, 1933, at par and accrued interest. They are due Oct. 15, 1938. If they could have been purchased for delivery on Oct. 15, 1920, just after the payment of the interest at a price of 85, what minimum rate of interest would they yield the investor?

10. The Cleveland Electric Illuminating Company issued 15 year 7 per cent bonds payable half-yearly, dated July 1, 1920, due July 1, 1935. These bonds are redeemable on any interest payment date, after the payment of the interest due, at 101 to July 1, 1921; at 102 to July 1, 1922; at 103 to July 1, 1923; thereafter to maturity at 107½. (*The Commercial and Financial Chronicle*, June 26, 1920.)

The price quoted for these bonds is 96, and it is claimed by a bond dealer

that they yield 7.45 per cent at this price. Check this statement by a calculation.

11. The Special School District No. 1 City of South St. Paul issued $475,000 of school bonds on July 1, 1920, at 6 per cent, payable semiannually. These bonds are available in denominations of $1000. They are to be redeemed in series, commencing July 1, 1932. If equal instalments are to be redeemed each half-year for 10 years to provide for the entire redemption, what should be the purchase price to yield 7 per cent, payable semiannually? To yield 5 per cent payable semiannually?

12. On July 15, 1920, Armour and Company issued 7 per cent 10 year gold notes, interest payable semiannually. These notes were offered to the public at 94.84, and were claimed by bond dealers to yield 7.75 per cent. Verify the accuracy of this statement about the yield.

13. A manufacturing company placed $30,000 of its undivided earnings in the hands of a broker to invest in 4 per cent government bonds redeemable in 9 years. The bonds cost 91 and commission $^1/_8$ per cent of face value of bonds. These bonds could be bought in denominations as low as $50. What was the face value of bonds bought? What uninvested amount was returned to the company? What interest did the company realize on the investment?

14. The Drake hotel in Chicago advertised first mortgage 6 per cent bonds for sale "at par and accrued interest to net 6 per cent from date of delivery." Interest coupons are payable April 1 and October 1. On June 1, a $1000 bond is delivered to a purchaser. What did the purchaser pay for it?

15. A father leaving an estate wills that it be converted into cash and invested for the benefit of his two children. The estate yielded $41,150. After paying the inheritance tax of 1½ per cent, the remainder was invested in as many $1000, 6 per cent bonds quoted 110 and redeemable in 15 years as the money would buy, the fractional part of $1000 being deposited in a 4 per cent savings bank. The guardian aims to replace the premium paid on the bonds by building up a sinking fund in the same savings bank. All interest is nominal and converted semiannually. What is the annual income of each child?

CHAPTER V

MATHEMATICS OF DEPRECIATION

62. Meaning of depreciation charges. In the operation of a business enterprise, there are losses which arise out of physical or functional deterioration of property consisting of buildings, machinery, and equipment. Part of the deterioration is handled by current repairs, but part of it is such that it cannot be satisfactorily provided for out of a repair fund. Thus, parts of a plant must be renewed and such renewal is likely to be an unfair charge on current income. The losses not cared for by current revenue are often called **depreciation,** and are handled in the accounts of the business by means of formal charges known as **depreciation charges.** For the purposes of the treatment in this book, depreciation charges will be understood to cover losses due to physical and functional decay, not including such losses as are made good by current repairs.

63. Different methods of treating depreciation. It would take us too far from our main purpose to present here arguments in support of a particular method of treating depreciation. That is more properly the province of the accountant and valuation engineer than of the student of mathematics. Our purpose is rather to present clearly a simple mathematical treatment of those methods which have sufficient mathematical content to be regarded as part of the mathematics of finance.

The methods which we shall consider are:

(1) The straight line method.
(2) Constant percentage of book value method.
(3) The sinking fund method.
(4) The "compound interest method."
(5) The unit cost method.

64. The straight line method. Under this method the annual depreciation charge is simply one of the equal annual

amounts which would reduce the book value of the article in its estimated life from its original value to its salvage value or scrap value. Thus, if C is the cost and S the scrap value, and n the estimated lifetime of the article, the annual depreciation under the straight line method is

$$\frac{C - S}{n}. \tag{1}$$

Sometimes the **wearing value** W of an article is defined as the original cost minus the scrap value. Thus, the annual depreciation charge is

$$\frac{W}{n}.$$

This method is called the straight line method, because as shown in Fig. 3, the graphical representation on coördinate paper, of either the book value or of the sum of depreciation charges for successive equal intervals, is a straight line.

Table A. Showing book values and depreciation charges at the end of any year of the life of a machine costing \$1000 with a salvage value of \$100 at the end of ten years.

(1) Age in Years	(2) Book Value at End of Year	(3) Depreciation Charges at End of Year	(4) Total Accumulation of Depreciation Charges
0	\$1000		
1	910	\$90	\$ 90
2	820	90	180
3	730	90	270
4	640	90	360
5	550	90	450
6	460	90	540
7	370	90	630
8	280	90	720
9	190	90	810
10	100	90	900

The straight line method would make the book values equal to the actual values if interest were neglected and if depreciation charges for equal times were equal.

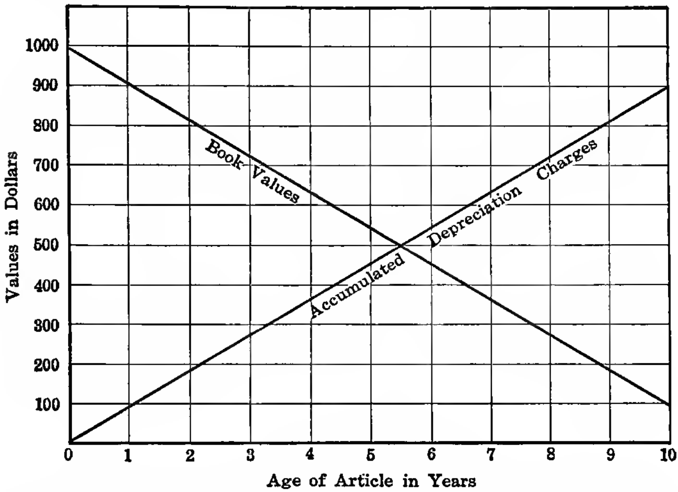

FIG. 3. Graphical illustration of book values and of accumulated depreciation charges—straight line method.

PROBLEMS

1. Show both by means of a tabular schedule and by a graph the accumulation of depreciation charges and the decline in book values when the straight line method is applied to a machine that cost \$100 and is to have a scrap value of \$20 at the end of 8 years.

2. A machine cost \$1000. Its estimated life is 20 years. The straight line method of depreciation is used. At the end of 10 years it was found that the original estimate was wrong, and that the machine would last only five more years. Give tabular and graphical representation of depreciation and decline in book value.

3. A telephone post set in earth cost \$8; one set in concrete cost \$12. The one set in earth lasts 12 years and the one set in concrete lasts 20 years. Give a tabular and graphical representation of the depreciation of each under the straight line method.

65. Constant percentage of book value method. The constant percentage of book value method charges off at the end

of each year a constant percentage of the book value at the beginning of the year.

This constant percentage must be determined so that at the end of a given number of years n, the book value of an article originally costing C will be reduced to the salvage value S.

Let x equal the percentage of the book value at the beginning of the year written off for the year to reduce C to S in n years.

Then
$$C(1 - x)^n = S,$$
and
$$(1 - x)^n = \frac{S}{C},$$
$$1 - x = \sqrt[n]{\frac{S}{C}},$$
$$x = 1 - \sqrt[n]{\frac{S}{C}}. \qquad (2)$$

The book value at the end of r years is then
$$C(1 - x)^r = \sqrt{\left(\frac{S}{C}\right)^r}. \qquad (3)$$

PROBLEMS

1. Find the constant per cent by which a value of \$1000 must be reduced at the end of each year for 10 years so that the scrap value is \$100.

It should be noted from (2) that, when $S = 0$, we have $x = 1$ for any assigned value of n. That is, in this case the book value would be reduced to zero at the end of 1 year no matter what we estimated n to be. This means simply that the method is impractical when the scrap value is zero. Furthermore, if the ratio of scrap value S to original cost C is very small, the depreciation charge is likely to be unreasonably large during the first years of operation.

Table B. Showing book values and depreciation charges at the end of any year of life of a machine costing $1000 with a value of $100 at the end of 10 years.

(1) Age in Years	(2) Book Value at End of Year	(3) Depreciation Charge for Year	(4) Total Accumulation of Depreciation Charges
0	$1000.		
1	794.33	$205.67	$205.67
2	630.96	163.37	369.04
3	501.19	129.77	498.81
4	398.11	103.08	601.89
5	316.23	81.88	683.77
6	251.19	65.04	748.81
7	199.53	51.66	800.47
8	158.49	41.04	841.51
9	125.89	32.60	874.11
10	100.00	25.89	900.00

The plan of decreasing the book values by a constant percentage tends towards relatively large depreciation charges during the early years of the life of the article, and small charges during the latter years of the life of the article.

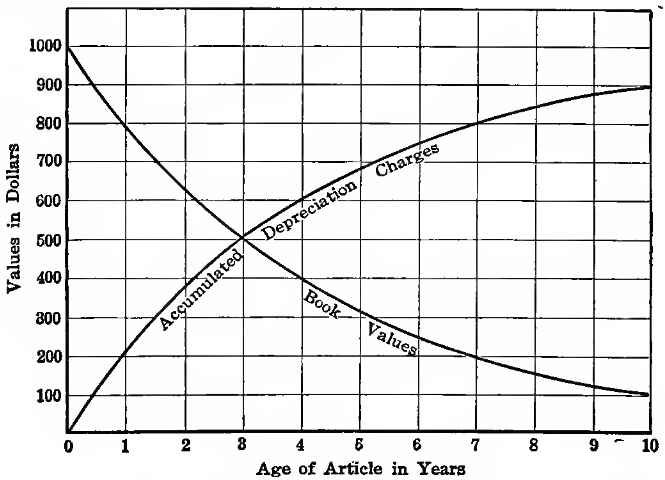

FIG. 4. Graphical illustration of book values and of accumulated depreciation charges—constant percentage of book value method.

Like the straight line method, this method totally disregards interest on the funds involved.

2. Show both by means of a tabular schedule and by a graph the accumulation of depreciation charges and the decrease in book values when the "constant percentage of book value method" is applied to a machine that cost \$1,200 and is to have a salvage value of \$200 at the end of 20 years.

3. Show by the constant percentage method both by means of a tabular schedule and by a graph the accumulation of depreciation charges and the decrease in book values of a machine costing \$1000 lasting for 10 years at the end of which time it has a junk value of \$10. Criticise the method for this case.

66. Sinking fund method. The sinking fund method of providing for depreciation makes the annual depreciation charge equal to the rent of an annuity certain which will accumulate in the lifetime of the article, at a given rate of compound interest, to the difference between cost and scrap values or to the wearing value. Thus, if C is the cost, S the scrap value, and n the estimated lifetime of an article, under the sinking fund method, the annual depreciation charge at the end of each of n years is

$$D = \frac{C - S}{s_{\overline{n}|}} = \frac{W}{s_{\overline{n}|}}, \tag{4}$$

where $s_{\overline{n}|}$ is the accumulated value of an annuity of 1 per year for n years at the assumed rate of interest. Under the sinking fund method, the funds to cover depreciation charges are usually invested in safe securities outside the business. The accumulated depreciation allowances at any time are the sum of the annual depreciation charges and interest on these at a prescribed rate.

The accumulation of depreciation allowances at the end of any year, say the rth year, is then given by

$$D_r = \frac{W s_{\overline{r}|}}{s_{\overline{n}|}}. \tag{5}$$

The wearing value W_r that still remains after r years of use is then given by

$$W_r = W - \frac{W s_{\overline{r}|}}{s_{\overline{n}|}}. \tag{6}$$

The **condition per cent** of an article at any time, say r years after it was new, is defined as the ratio of the remaining wearing value W_r to the original wearing value W. Hence, the condition per cent is

$$\frac{W_r}{W} = 1 - \frac{s_{\overline{r}|}}{s_{\overline{n}|}}. \tag{7}$$

Table C. Showing under the sinking fund method the book values, annual depreciation charges, and accumulations in the sinking fund at the end of any year of life of a machine costing \$1000 with salvage value zero at end of 10 years. Interest at five per cent.

(1) Age in Years	(2) Book Value at End of Year	(3) Annual Payment to Sinking Fund to Cover Depreciation	(4) Interest on Depreciation Allowances	(5) Total in Sinking Fund
0	\$1000 *			
1	920.495	\$79.5046	\$ 0.	\$ 79.505*
2	837.016	79.5046	3.9752	162.984
3	749.362	79.5046	8.1492	250.638
4	657.325	79.5046	12.5319	342.675
5	560.687	79.5046	17.1337	439.313
6	459.217	79.5046	21.9656	540.783
7	352.673	79.5046	27.0392	647.327
8	240.802	79.5046	32.3664	759.198
9	123.338	79.5046	37.9599	876.662
10	0.000	79.5046	43.8331	1000.000

* These numbers were calculated by using one more place than is recorded in this column.

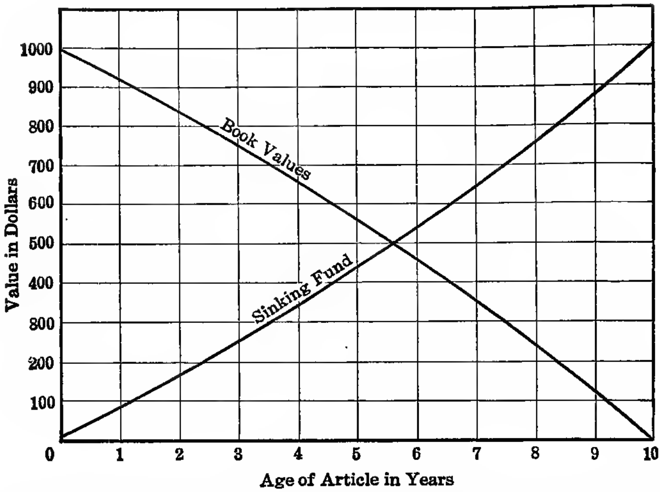

FIG. 5. Graphical illustration of book values and of accumulations in sinking fund to cover depreciation—sinking fund method.

PROBLEMS

1. By the use of formula (5), verify the totals in the sinking fund as given in column (5) in Table C at the ends of the fourth and eighth years.

2. An automobile has an original value of $1750. Its depreciation is to be covered by a sinking fund at six per cent under the conditions that the value at the end of eight years will be $250. Find the book value at the end of five years. *Ans.* $895.68.

3. A machine has an original value of $2000. Its depreciation is to be covered by a sinking fund at five per cent under the conditions that the scrap value at the end of ten years will be $200. What wearing value remains at the end of six years? What is the condition per cent at the end of six years? *Ans.* $826.59; 45.92 per cent.

67. Compound interest method. The so-called "compound interest method" of providing for depreciation has been proposed by the Special Committee of the American Society of Civil Engineers to Formulate Principles and Methods for the Valuation of Railroad Property and other Public Utilities.*

* For final report of committee, see *Transactions Amer. Soc. of Civil Engineers,* Vol. LXXXI, pp. 1311–1620, Dec., 1917.

In this method, the sinking fund principle is applied but the funds are used in the business for additions to property instead of investing them in outside securities. Attention is directed both to the interest necessary to give a fair return on the investment and to the interest involved in amortizing the book value of the property.

To develop the compound interest method, let

C = original value of the article concerned,

S = scrap value,

i = rate of interest for a fair return on investment,

i' = rate of interest used in amortizing book values,

n = number of years of service life of an article.

Under the sinking fund method it would be necessary to set aside yearly for depreciation

$$\frac{C - S}{s_{\overline{n}|}}$$

at rate i', and this is also the depreciation charge for the first year under the compound interest method. For any later year, say for the tth year, under the compound interest method, the depreciation charge would be

$$(1 + i')^{t-1} \frac{C - S}{s_{\overline{n}|}} \text{ at rate } i'.$$

That is, we may state the principle involved in this method by saying that the depreciation for any year is equal to the equated value (Art. 29) for any other year at rate i'.

Then clearly at the end of the first year, there should be available

$$Ci$$

for a fair return on the investment and

$$\frac{C - S}{s_{\overline{n}|}},$$

where $s_{\overline{n}|}$ is at rate i' for depreciation. Thus the combined sum available at the end of the first year should be

$$Ci + \frac{C - S}{s_{\overline{n}|}} \text{ at rate } i'.$$

At the end of second year the sum available is

$$\left(C - \frac{C - S}{s_{\overline{n}|}} \text{ at rate } i'\right) i + \frac{C - S}{s_{\overline{n}|}} (1 + i') \text{ at rate } i',$$

and so on.

The progress of funds under this method may well be followed by the preparation of schedules such as that in Table D where column (3) gives the values of $\dfrac{C - S}{s_{\overline{n}|}} (1 + i')^{t-1}$ for $C =$ 1000, $S = 0$, $i' = .05$ and t takes on the values from 1 to 10.

It may be noted that when the rates of interest on the book value of the investment and on the depreciation allowance are equal, the effect of the "compound interest method" on available revenue * is just the same as under the sinking fund method. The difference in this case would consist only in a difference of form in making entries in accounts.

It will be noted that columns (2) and (4) of Table D exhibiting a schedule under the compound interest method are the same as columns (2) and (5) of Table C exhibiting the corresponding schedule under the sinking fund method, and that each number in column (3) of Table D is simply the number immediately above it plus 5 per cent of that number.

The depreciation allowance for the first year is

$$\frac{1000}{s_{\overline{10}|} \text{ (at 5 per cent)}} = 79.5046,$$

and it is increased by 5 per cent each year of the service life of the article.

If the interest rate for a fair return on the investment is taken equal to the rate assumed on depreciation allowances, the combined annual return on depreciation allowances and on investment is constant as illustrated by column (7) of Table D. If the interest rate for a fair return on the investment is taken higher than the rate assumed on depreciation allowances, the

* See Paton and Stevenson, *Principles of Accounting*, p. 519, 1918

combined annual return on depreciation allowances and on investment decreases throughout the service life of the article as illustrated by column (8) of Table D.

With respect to the principles on which the compound interest method is based, it seems clear that there is involved the tacit assumption that the depreciation is a function of the interest rate. It seems doubtful whether there is good ground for such an assumption.

Since the book values in Table D and the total of depreciation allowances correspond to values found under the sinking fund method, the graphs are the same as those given in Figure 5.

We give, however, in Figure 6 the graphs of values in columns (7) and (8) of Table D. These are the combined yearly returns using 5 per cent on depreciation allowances, and 5 per cent to form column (7) and 8 per cent to form column (8) for returns on investment or book values. It should be noted that the vertical scale in Fig. 6, is ten times the vertical scale in Figs. 3, 4, 5.

Table D. Showing under the compound interest method the annual depreciation charges—interest at 5 per cent on depreciation allowances and at 5 and 8 per cent on book values. Status at end of each year of life of a machine costing $1000 with a value zero at the end of 10 years.

(1) Age in Years (end)	(2) Book Value at End of Year	(3) Depreciation Allowance of the Year (end)	(4) Total of Depreciation Allowances to End of Year	Return of Year = Remaining Book Value × Rate on Book Value		Combined Depreciation Allowance and Return on Investment for the Year	
				(5) 5 per cent	(6) 8 per cent	(7) Return at 5 per cent (7) = (3) + (5)	(8) Return at 8 per cent (8) = (3) + (6)
0	$1000.						
1	920.495*	79.5046	$79.505*	$50.0000	$80.0000	$129.5046	$159.5046
2	837.016	83.4798	162.984	46.0248	73.6396	129.5046	157.1194
3	749.362	87.6538	250.638	41.8508	66.9612	129.5046	154.6150
4	657.325	92.0365	342.675	37.4681	59.9489	129.5046	151.9854
5	560.687	96.6383	439.313	32.8663	52.5860	129.5046	149.2243
6	459.217	101.4702	540.783	28.0344	44.8550	129.5046	146.3252
7	352.673	106.5437	647.327	22.9608	36.7373	129.5045	143.2810
8	240.802	111.8709	759.198	17.6337	28.2138	129.5046	140.0847
9	123.338	117.4644	876.662	12.0401	19.2642	129.5045	136.7286
10	0.000	123.3376	1000.00	6.1669	9.8670	129.5045	133·2046

* See footnote, p. 113.

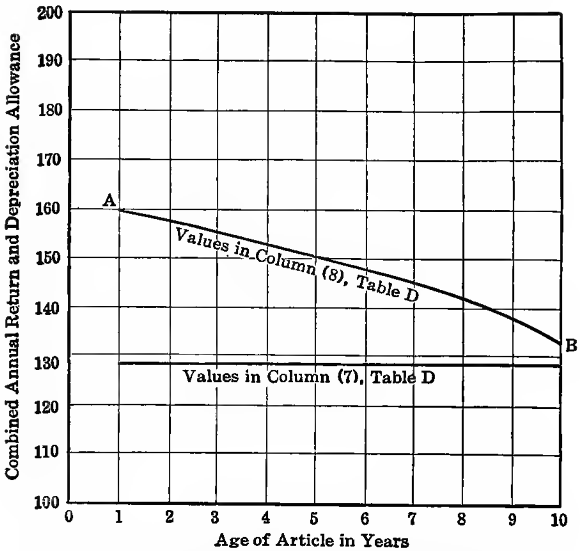

FIG. 6. Graphical illustration of combined yearly returns on book values and depreciation allowances.

PROBLEM. Show by means of a tabular schedule the operation of book values, depreciation, and return on investment when the compound interest method is applied to a machine which cost \$100 and has a value of \$20 at the end of 8 years. Assume 10 per cent as a fair return on investment, and 5 per cent for the rate on depreciation allowances.

68. The Unit Cost Method.* None of the calculations for depreciation given thus far have taken into account explicitly the important question of improvements in machinery, or the decrease in efficiency of the old machine. The conception that underlies the unit cost method is that the value of a plant should be decreased from year to year to such an extent that the net cost of a unit of the output should be the same as for a machine that could be found to replace the given machine. That is to say, the old machine should be so valued that its per unit

* *Transactions American Society of Civil Engineers*, Dec., 1917; Saliers, E. A., *Principles of Depreciation*, pp. 166–171; Kester, R. B., *Accounting Theory and Practice*, Vol. II, pp. 167–168.

cost of output, taking account of all charges for repairs, depreciation, interest, and operating expenses, should be the same as for a new machine. This seems so sound in principle that we may well study the mathematical formulation of the problem. The factors to be taken into account are: (1) The values of old and new machines, (2) the annual operating expenses of both machines for their lifetimes, not including repairs, (3) the annuities sufficient to accumulate funds for repairs, (4) the number of units of output of each machine.

Let C = the first cost of new machine,
N = lifetime of new machine,
F = annual rent of annuity to amortize C in N years,
O = annual operating expense of new machine excluding repairs,
R = annual rent of annuity for repairs on machine,
Y = number of units of output per year.

Let the corresponding lower case letters f, o, r, and y denote the corresponding quantities for the old machine, and let c be the value of the old machine at the time of making the valuation, and n the remaining lifetime of the old machine. Let i be the rate of interest.

Then the cost per unit output of the new machine is

$$\frac{O + R + F + Ci}{Y},$$

and of the old machine

$$\frac{o + r + f + ci}{y}.$$

From the basic principle that unit cost is to be the same for the two machines, we have

$$\frac{O + R + F + Ci}{Y} = \frac{o + r + f + ci}{y}, \tag{8}$$

The annuity of annual rent F necessary to accumulate to C in N years is given by

$$F = \frac{C}{s_{\overline{N|}}} = CX, \tag{9}$$

where
$$X = \frac{1}{s_{\overline{N}|}}.$$

Similarly,
$$f = \frac{c}{s_{\overline{n}|}} = cx, \tag{10}$$

where
$$x = \frac{1}{s_{\overline{n}|}}.$$

Substitution of $F = CX$ and $f = cx$ in (8) gives

$$\frac{O + R + CX + Ci}{Y} = \frac{o + r + cx + ci}{y}. \tag{11}$$

Solving this equation for c, we obtain

$$c = \frac{y}{x + i}\left(\frac{O + R + XC + iC}{Y}\right) - \frac{o + r}{x + i},$$

$$= \frac{y}{x + i}\left(\frac{O + R + XC + iC}{Y} - \frac{o + r}{y}\right), \tag{12}$$

$$= ya_{\overline{n}|}\left(\frac{O + R + XC + iC}{Y} - \frac{o + r}{y}\right), \tag{13}$$

since
$$x + i = \frac{1}{s_{\overline{n}|}} + i = \frac{1}{a_{\overline{n}|}} \text{ by Art. 41.}$$

If the annual number of units of output of the old and new machines are the same,

$$Y = y,$$

then (13) becomes

$$c = a_{\overline{n}|}\,(O + R + XC + iC - o - r). \tag{14}$$

But
$$X + i = \frac{1}{s_{\overline{N}|}} + i = \frac{1}{a_{\overline{N}|}}.$$

Hence,
$$c = a_{\overline{n}|}\left[\frac{C}{a_{\overline{N}|}} + O + R - o - r\right]. \tag{15}$$

When we have not only $Y = y$, but the annual operating expense equal so that $O = o$, then

$$c = a_{\overline{n}|}\left[\frac{C}{a_{\overline{N}|}} + R - r\right], \tag{16}$$

If when $Y = y$, we have

$$O + R = o + r,$$

then from (15),

$$c = \frac{Ca_{\overline{n}|}}{a_{\overline{N}|}}. \tag{17}$$

PROBLEMS

1. A machine with a remaining service life of 5 years turns out 20 units of work per year. Its operation costs \$250 per year, and repairs \$200 per year. A new machine that turns out 25 units costs \$1200. It has a probable life of 12 years, will cost \$300 per year for operation and \$200 for repairs. Find the value of the old machine on the basis of interest at 5 per cent.

SOLUTION: $C = \$1200$, $N = 12$, $O = \$300$, $R = \$200$, $Y = 25$, $n = 5$, $o = 250$, $r = 200$, $y = 20$.

$$X = \frac{1}{s_{\overline{12}|}} = .062825, \quad a_{\overline{n}|} = a_{\overline{5}|} = 4.32948.$$

Substitute in (13) and we have

$$c = 20\,(4.32948)\left[\frac{300+200+1200\,(.062825)+1200\,(.05)}{25} - \frac{250+200}{20}\right].$$

$$= 86.5896\,[25.4156 - 22.50] = \$252.46.$$

2. If the new machine described in problem 1 had been capable of turning out 40 units of work, what would have been the value of the old machine? Interpret the result.

3. How many units of work must be turned out by the new machine in problem 1, so that the old machine would be worthless?

4. Derive a formula for the number of units a new machine should turn out in order to make the value of the old machine zero.

5. A factory has a machine that cost \$1000 and has an estimated life-time of 25 years, turning out 50 units of production per year. The operation of the machine is to cost on the average \$400 per year, and the rent of the annuity to provide for repairs is \$200. When the machine is 10 years old, it is found that improvements have been made so that a machine

costing $1000 with a lifetime of 25 years will have an output of 60 units, and can be operated for $500 per year and with repair allowances of $200. Find the value of the old machine at the end of 10 years of use by the unit cost method if money is worth 5 per cent.

6. Find the value of the old machine in problem 5 if the cost of operation had remained at $400.

69. Renewal of $\frac{1}{n}$ th of a plant each year after it reaches full size by equal annual expansion for n years.* Thus far in our treatment of depreciation, we have assumed that the article or plant involved is put into use all at once. Interesting and important problems present themselves when we modify this assumption in certain ways. Thus, we shall consider the case in which $\frac{1}{n}$ th part of a plant is constructed each year for n years and shall determine under these conditions the amount accumulated at interest rate i in a sinking fund at the end of any number of years m.

Let c = the contribution to the sinking fund at the end of each year for each unit of investment during the year. Then clearly in order that 1 may be available for renewals at the end of n years, it is necessary (Art. 41), that

$$c = \frac{1}{s_{\overline{n|}}}.$$

Hence, corresponding to an investment of 1 the first year, the accumulation at the end of the mth year ($m \overline{\overline{<}} n$), before anything has been paid out of the sinking fund, is

$$c \cdot s_{\overline{m|}} = \frac{1}{s_{\overline{n|}}} \quad s_{\overline{m|}}. \tag{18}$$

For each investment of 1 the second year, such accumulation at the end of the mth year is

$$c \cdot s_{\overline{m-1|}} = \frac{1}{s_{\overline{n|}}} \cdot s_{\overline{m-1|}}. \tag{19}$$

* See Grunsky and Grunsky, *Valuation, Depreciation, and the Rate-Base,* pp. 90–94.

Similarly, for investment of 1 the third, fourth, fifth, and mth years, the corresponding accumulations are

$$cs_{\overline{m-2}|},\ cs_{\overline{m-3}|},\ \ldots,\ c \text{ or}$$

$$\frac{s_{\overline{m-2}|}}{s_{\overline{n}|}},\ \frac{s_{\overline{m-3}|}}{s_{\overline{n}|}},\ \ldots,\ \frac{1}{s_{\overline{n}|}}. \tag{20}$$

Hence, the total amount S_m in the sinking fund at the end of m years is

$$S_m = \frac{1}{s_{\overline{n}|}}\left(s_{\overline{m}|} + s_{\overline{m-1}|} + s_{\overline{m-2}|} + \ \ldots + s_{\overline{1}|}\right). \tag{21}$$

Expressing the terms in the parenthesis in terms of i and m by formula (1), Art. 32, we have from (21),

$$S_m = \frac{1}{s_{\overline{n}|}}\left[\frac{(1+i)^m - 1}{i} + \frac{(1+i)^{m-1} - 1}{i} + \frac{(1+i)^{m-2} - 1}{i} + \ldots + \frac{1+i-1}{i}\right]$$

$$= \frac{1}{is_{\overline{n}|}}\left[(1+i)^m + (1+i)^{m-1} + \ldots + (1+i) - m\right] \tag{22}$$

$$= \frac{1}{is_{\overline{n}|}}\left[\frac{(1+i)^{m+1} - (1+i)}{i} - m\right], \tag{23}$$

$$= \frac{1+i}{is_{\overline{n}|}}\left[\frac{(1+i)^m - 1}{i} - \frac{m}{1+i}\right], \tag{24}$$

$$= \frac{1+i}{is_{\overline{n}|}}\left(s_{\overline{m}|} - \frac{m}{1+i}\right). \tag{25}$$

This formula (25) holds for $m < n$, and for $m = n$ just before deducting the 1 that is to be taken from the fund at the end of n years for renewals. After deducting the 1 for renewals at the end of the nth year, the amount S'_n in the sinking fund is

$$S'_n = \frac{1+i}{is_{\overline{n}|}}\left(s_{\overline{n}|} - \frac{n}{1+i}\right) - 1 = \frac{1}{i} - \frac{n}{is_{\overline{n}|}}. \tag{26}$$

With no further extensions of the plant after n years, we may regard the situation as in a stable state with respect to renewals, so that the amount S'_m in the sinking fund at the end of any year when $m > n$, and after deductions of 1 for renewals, is the same as S'_n.

From another point of view, we must have in the sinking fund at the end of m years from the beginning, $(m>n)$, the compound amount of S'_n for $m-n$ years diminished by the amount of an annuity of rent

$$1 - nc = 1 - \frac{n}{s_{\overline{n}|}}$$

for $m-n$ years. This gives from (26)

$$\left(\frac{1}{i} - \frac{n}{is_{\overline{n}|}}\right)(1+i)^{m-n} - \left(1 - \frac{n}{s_{\overline{n}|}}\right)s_{\overline{m-n}|}. \tag{27}$$

This expression simplifies into

$$\frac{1}{i} - \frac{n}{is_{\overline{n}|}}.$$

This reduction is left as an exercise for the student.

PROBLEMS

1. A plant is constructed by putting \$10,000 into machinery each year for 10 years. Then one-tenth of the plant is renewed yearly. The depreciation is handled by a sinking fund method at 5 per cent interest. Find the amount in the sinking fund (1) at the end of five years, (2) at the end of ten years after the first allotment for renewals, (3) at the end of fifteen years after the sixth allotment for renewals. *Ans.* \$12,751.15; \$40,990.80; \$40,990.80.

In the expansion and renewal methods described in this article, the following interesting question arises: What per cent of the investment is the accumulated amount in the sinking fund? This depends both on the rate of interest i and on the value of n.

For example, if $n = 10$ and $i = .06$, the amount in the sinking fund at the end of 7 years is

$$\frac{1.06}{.06\, s_{\overline{10}|}}\left[s_{\overline{7}|} - \frac{7}{1.06}\right] = 1.3403\,[8.3938 - 6.6038] = 2.399$$

$$= 34.3 \text{ per cent of } 7.$$

If $n = 10$, and $i = .06$, we have from (26) for the amount in the sinking fund at the end of 10 years, just after deducting 1 for renewals,

$$S'_n = \frac{1}{.06} - \frac{10}{(.06)(13.181)} = 4.02 = 40.2 \text{ per cent of } 10.$$

Hence, 40.2 per cent is the answer to our question.

2. Fill out the following schedule when $\frac{1}{n}$ of a plant is to be renewed yearly after it reaches full size by equal annual expansion for n years, showing amounts * in sinking fund expressed in percentages of the book values of the investment.

At End of Year	$n=5$ years		$n=10$ years		$n=20$ years		$n=30$ years	
	3%	6%	3%	6%	3%	6%	3%	6%
1	18.8	17.7						
5	38.8*	37.7						
10	38.8	37.7						
15								
20								
30								
40								

3. In starting a transfer business it is planned to purchase 10 cabs annually for 8 years at a cost of $1000 per cab. It is estimated that 8 years is the service life of such a cab. It is further planned to replace the worn out cabs by the sinking fund method described above. Find the amount in the sinking fund at the end of 8 years at 5 per cent, just after the first allotment for replacements.

70. Depreciation of a mining property. Investment in a mine that is being worked should yield not only interest on the investment, but additional income to provide for the restoration of the original capital when the mineral is exhausted. The sinking fund created in this connection for restoring the original capital is often called a **redemption fund.** Thus, if the net income from a mine may be taken from the estimates of the mining

* Amounts are to be given after allotments for renewals have been made.

engineer to be R at the end of each year for n years, then the present value P of the mine is

$$P = Ra_{\overline{n}|} = R\frac{1 - v^n}{i}. \tag{28}$$

If this value is to yield a rate i, the interest per year is

$$R(1 - v^n).$$

The excess of the income R over the interest is then

$$R - R(1 - v^n) = Rv^n. \tag{29}$$

This is the amount that should be put into a redemption fund annually at rate i to restore the original capital.

If the conditions of investment make desirable a different rate of interest, usually a higher rate, on the investment than the rate i assumed for accumulating the sinking fund, the problem is still capable of simple solution. Let i' be the rate of interest on the investment. Then

$$R - Pi'$$

is annually paid into the sinking fund. The redemption fund accumulated in n years should be equal to the original value of the property. That is,

$$(R - Pi')\, s_{\overline{n}|} = P, \tag{30}$$

$$P + Pi'\, s_{\overline{n}|} = Rs_{\overline{n}|},$$

Hence,
$$P = \frac{Rs_{\overline{n}|}}{1 + i'\, s_{\overline{n}|}} = \frac{R}{\dfrac{1}{s_{\overline{n}|}} + i'}. \tag{31}$$

where $s_{\overline{n}|}$ is at rate i.

PROBLEMS

1. A mine will yield a net annual income of \$25,000 for 20 years. If the income is assumed to be paid in one sum at the end of the year, what is the value of the mine if money is worth 5 per cent? What is its value if 10 per cent is the rate stipulated for this class of investment and 5 per cent is the rate on the redemption fund? *Ans.* \$311,555.26, \$191,949.50.

2. In the Appalachian oil fields, an oil property is to be valued on the basis of a production of 1000 barrels per day for five years at a net value of $1.00 per barrel. The valuation is to be such as to permit the purchaser to recover his capital at the end of five years with 5 per cent on the redemption fund and 12 per cent on the investment. Find the value of the property assuming 365 days per year and that the income of any year is available to the owner at the end of the year.

3. The annual profits of a business are $1000 less a certain amount paid into a sinking fund planned to amount in 20 years at 4 per cent to the capital invested. After this reduction the net profits were found to yield 7 per cent on the capital. What was the capital invested? *Ans.* $9654.14.

4. The Miami Copper company has an ore body estimated at not less than 14,000,000 tons of ore averaging 56 pounds of copper to the ton. Of this 80 per cent can be brought to the market at a cost of 9 cents a pound. The company which is capitalized at 600,000 shares is said to have enough money in the treasury to bring it to the point of production at this rate. The deposit is worked at the rate of 700,000 tons a year. Of this, the marketed part sells at 15 cents a pound. Interest on the investment is to be 7 per cent after providing for a sinking fund to be invested at 4 per cent. What is the yearly profit per share before providing for the sinking fund? What yearly sum per share is set aside for the sinking fund? What is the value of a share? (From Finlay's *Cost of Mining.*) *Ans.* $3.136; $1.017; $30.26.

71. Capitalized cost of an article. The expression **capitalized cost** * of an article is sometimes used for the first cost plus the present value of perpetual renewals. In thinking out a permanent system of finance, the conception of the capitalized cost of an article is important in connection with depreciation and renewals. If C is the original cost of an article and an additional amount C must be expended every n years for renewals, the capitalized cost of the article is defined to be

$$C + \frac{C}{is_{\overline{n}|}}, \tag{32}$$

because $\dfrac{C}{is_{\overline{n}|}}$ is the present value of renewals every n years ad infinitum. (See (43) Art. 42.) The capitalized cost given by (32) can be expressed in another form as follows:

* Compare Art. 42.

$$C + \frac{C}{is_{\overline{n}|}} = \frac{C}{i}\left(i + \frac{1}{s_{\overline{n}|}}\right),$$

$$= \frac{C}{i} \cdot \frac{1}{a_{\overline{n}|}}. \tag{33}$$

PROBLEMS

1. An automobile costing \$2000 will be worn out in five years. What sum will be required not only to purchase the machine but to provide for getting a new machine each five years on the supposition that the cost of the machine will remain constant, and money is worth 5 per cent? *Ans.* \$9238.99.

2. What sum would be required for the situation described in problem 1 if at the end of 5 years of wear the automobile had a secondhand value of \$500? *Ans.* \$7429.24.

3. What is the capitalized cost of a bridge built for \$40,000 if it is to be renewed every 40 years and if money is worth 5 per cent?

72. Investments to extend service life of an article. It is a problem of considerable importance to determine the amount that may profitably be expended on an article to extend its service life. To treat this problem, let

C = original cost of an article to last n years. Its capitalized cost, by (33) Art. 71, is

$$\frac{C}{i} \cdot \frac{1}{a_{\overline{n}|}}. \tag{34}$$

Let $C + y$ be the cost of an article that will have a service life of $n + x$ years.

Its capitalized cost is

$$\frac{(C + y)}{i} \cdot \frac{1}{a_{\overline{n+x}|}}. \tag{35}$$

If the use of the two articles is to be equally economical in the long run, their capitalized costs must be equal. That is,

$$\frac{(C + y)}{i} \cdot \frac{1}{a_{\overline{n+x}|}} = \frac{C}{i} \cdot \frac{1}{a_{\overline{n}|}}. \tag{36}$$

Solving this equation for y, we have

$$y = C\frac{a_{\overline{n+x}|} - a_{\overline{n}|}}{a_{\overline{n}|}}, \tag{37}$$

$$= C\frac{v^n - v^{n+x}}{ia_{\overline{n}|}}, \tag{38}$$

$$= \frac{Cv^n\left[\frac{1-v^x}{i}\right]}{a_{\overline{n}|}},$$

$$= \frac{Cv^n a_{\overline{x}|}}{a_{\overline{n}|}}, \tag{39}$$

$$= \frac{Ca_{\overline{x}|}}{s_{\overline{n}|}}, \tag{40}$$

since $a_{\overline{n}|} = v^n s_{\overline{n}|}$.

PROBLEMS

1. A farmer learns that a certain kind of fence post that costs 40 cents each will last 10 years. What can he afford to expend per post for treatment that will double the service life of the posts if money is worth 5 per cent? *Ans.* 24.6 cents.

2. What amount would a railroad company be justified expending per tie to extend the life of ties costing \$1.25 each from 10 to 18 years money being worth 4 per cent?

73. Composite life of a plant. The **composite life of a plant** may be described as a sort of average lifetime of the several parts, and it is defined more precisely as the time required for the total equal annual depreciation charges for the plant to accumulate at an assumed rate of interest to the total wearing value of the plant.

Let $W_1, W_2, \ldots, W_t$ be the wearing values of the several parts, with service lives

$$n_1, n_2, \ldots, n_t$$

respectively and let $W = W_1 + W_2 + \ldots + W_t$ be the wearing value of the whole.
Also let $D_1, D_2, \ldots, D_t$ be the annual depreciation charges of the several parts and

$$D = D_1 + D_2 + \ldots + D_t,$$

be the depreciation of the whole. If the assumed rate of interest were zero, under the straight line method the composite life n is clearly given by

$$n = \frac{W}{D}. \tag{41}$$

But with a rate of interest i $(i > 0)$, the composite life is the value of n determined from the equation

$$D\, s_{\overline{n|}} = W. \tag{42}$$

To solve (42) for n, replace $s_{\overline{n|}}$ by

$$\frac{(1+i)^n - 1}{i},$$

and divide through by D.

This gives

$$\frac{(1+i)^n - 1}{i} = \frac{W}{D},$$

$$(1+i)^n - 1 = \frac{Wi}{D},$$

$$(1+i)^n = 1 + \frac{Wi}{D}, \tag{43}$$

$$n \log (1+i) = \log \left(1 + \frac{Wi}{D}\right),$$

$$n = \frac{\log \left(1 + \frac{Wi}{D}\right)}{\log (1+i)}. \tag{44}$$

An approximate value of n may be obtained as follows:
From (42) and the definitions of D and W, we have

$$(D_1 + D_2 + \ldots + D_t)\, s_{\overline{n|}} = W_1 + W_2 + \ldots + W_t. \tag{45}$$

But
$$D_1 = \frac{W_1}{s_{\overline{n_1}|}},$$
$$D_2 = \frac{W_2}{s_{\overline{n_2}|}},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$D_t = \frac{W_t}{s_{\overline{n_t}|}},$$

where
$$s_{\overline{n_1}|} = \frac{(1+i)^{n_1} - 1}{i} = \frac{1 + n_1 i + \dots - 1}{i},$$
$$= n_1 \text{ for a first approximation.}$$

Hence,
$$D_1 = \frac{W_1}{n_1}, \text{ approximately,}$$
$$D_2 = \frac{W_2}{n_2}, \text{ approximately,}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$D_t = \frac{W_t}{n_t}, \text{ approximately.}$$

Substitution of these approximate values in (45) gives
$$\left(\frac{W_1}{n_1} + \frac{W_2}{n_2} + \dots + \frac{W_t}{n_t}\right) n = W_1 + W_2 + \dots + W_t.$$

Hence,
$$n = \frac{W_1 + W_2 + \dots + W_t}{\dfrac{W_1}{n_1} + \dfrac{W_2}{n_2} + \dots + \dfrac{W_t}{n_t}}. \tag{46}$$

This approximate value of n is thus simply the harmonic mean * of n_1 taken W_1 times, n_2 taken W_2 times, and so on to n_t taken W_t times.

* The harmonic mean of a series of numbers is the reciprocal of the arithmetic mean (ordinary average) of their reciprocals.

PROBLEMS

1. Find by the exact formula (44) the composite life of a plant consisting of the following parts:

(*a*) A part costing when new $5000 with a scrap value $200 after 10 years of use.

(*b*) A part costing when new $3000 with a scrap value $300 after 15 years of use.

(*c*) A part costing when new $10,000 with a scrap value $500 after 20 years of use.

(*d*) A part costing when new $12,000 with a scrap value $1000 after 12 years of use.

Assume money worth 5 per cent.

2. Find the approximate composite life of the plant described in problem 1 by the formula (46) and compare the result with that of problem 1.

MISCELLANEOUS PROBLEMS

1. Find the constant per cent by which a book value of $1200 must be reduced each year for 15 years so as to have remaining a value of only $200. *Ans.* 11.26 per cent.

2. An automobile cost when new $4000, and the depreciation is to be covered by the sinking fund method at 5 per cent on the assumption that the scrap value of the machine will be $600 at the end of 5 years. What wearing value still remains at the end of the 3rd year and what is the condition per cent of the machine?

3. Make a schedule to show, for each year of life of a machine costing $4000, the operation of the sinking fund method of covering depreciation when the life of the machine is to be 12 years with a scrap value zero, and when the interest rate is taken at 6 per cent.

4. A brick plant has a press that cost when new $2000 and whose estimated service life is 20 years turning out 2500 units of production per year. The operation of the machine is to cost on the average $2400 per year, the rent of the annuity to provide for repairs is $100. When the machine has been used 8 years, it turns out that on account of improvements a machine costing $2500 with a lifetime of 20 years will have an output of 5000 units of production per year. The cost of operation and the repair allowance are the same as for the old machine. Find by the unit cost method, the value of the old machine at the end of 8 years of use, money being worth 5 per cent.

5. A building just completed cost $300,000. It is estimated that $1000 will be required at the end of each year for repairs, and that every ten years there must be renovation to the extent of $12,000, and that with

such care the building will have a service life of 40 years with a salvage value of $10,000. Find what equal annual sum should be set aside at 5 per cent interest to cover repairs, renovations, and replacements.

6. A man wishes to donate a building to cost $200,000 and to provide for its rebuilding every 50 years at the same cost. What amount should he donate if the sums for rebuilding are invested at 4 per cent? *Ans.* $232,751.

7. In starting an automobile business, it is planned to put $20,000 into equipment each year for 10 years. It is estimated that 10 years is the service life of the equipment. The plan is to replace equipment worn out by the sinking fund method. Find the amount in the sinking fund at the end of 10 years at 4 per cent just after the first allotment for replacing equipment.

8. The net annual income of a mine is estimated at $50,000 for 25 years. What is its value if money is worth 4 per cent? What is its value if 12 per cent is the rate stipulated on this class of investment, and 4 per cent is the rate on the redemption fund?

9. What is the capitalized cost of the street pavement to middle of street in front of a lot 50 feet wide when the initial cost of paving is $300 and when it must be renewed every 10 years? Assume money worth 5 per cent.

10. The owner of the lot described in problem 9 found that the first paving of a street lasted only 7 years, when the cost of paving was $300. How much additional could he afford to pay if money is worth 5 per cent to get a better pavement with a service life of 15 years? If money is worth 4 per cent? 6 per cent? *Ans.* $238.14; $255.73; $221.94.

11. An industrial plant has purchased a machine for $5000. The machine is estimated to depreciate to a value of $1000 in 12 years. By the sinking fund method with 5 per cent interest, present a schedule of book values and of accumulations of sinking fund to cover depreciation.

12. An industrial plant has a building that cost $30,000 which is to be written entirely off the books by the sinking fund method at 4 per cent in 12 years. Show schedule.

13. A water plant is built by a public-utility corporation at a cost of $500,000. It is estimated that the plant and mains will give satisfactory service for 20 years, at which time it will be necessary practically to rebuild the property. Make a tabular schedule showing the decrease in book value and the accumulations under the sinking fund method at 5 per cent.

14. For problem 13, make a tabular schedule under the "compound interest method," when the depreciation allowance is to be at 5 per cent

interest, but the fair return on the investment is to be at 8 per cent. Include in the schedule for each year the combined addition to depreciation allowance and return on investment for the year.

15. Determine the composite life and annual depreciation charge of a plant composed of the following parts:

	Value	Estimated Life	Scrap Value
Buildings	$100,000	50 years	$2000.00
Machinery	45,000	20 years	4500.00
Tools	10,000	5 years	200.00
Patterns	15,000	4 years	300.00

Depreciation is to be calculated by the sinking fund method and money is worth 5 per cent.

16. A steamboat having a probable life of 30 years yields on the average an income of 8 per cent of its original cost when nothing is set aside for replacement of the boat when worn out. If money can be set aside at 5 per cent in a sinking fund to replace the steamboat when worn out, what effective rate is earned on the investment? *Ans.* 6.4949 per cent.

17. Telephone poles in soil last 12 years, in concrete 20 years. If a telephone pole set in soil costs $6, what can the company afford for concrete work if money can be invested at 4½ per cent?

CHAPTER VI

THE OPERATION OF FUNDS IN BUILDING AND LOAN ASSOCIATIONS

74. Fundamental principles. The primary purpose of a building and loan association is to provide funds to be invested in loans, upon sufficient security, to those members or shareholders who need funds to build homes.

There are, in general, some participants of the association who are investors only, and there are others who are both investors and borrowers. The borrowers are investors in that they are investing in stock to be used in extinguishing their indebtedness.

Various plans are in operation in different associations with respect to accepting funds and issuing stock. Some of them will accept lump sums with limitations on the amount. It is the usual custom for all of them to accept regular monthly payments. In fact, most of them encourage the monthly payment plan as a scheme of saving.

To illustrate the monthly payment plan, let us take first the case of an investor who subscribes for a share of $100 on which he pays instalments, say $1 per month, to the association. The total of these instalments plus his proportionate part of the profits in time amounts to $100, when the stock is said to **mature** and he is entitled to $100. In case of a borrower of $100, say in an association operating on a 7 per cent nominal interest basis, he pays the association monthly in advance for interest

$$\frac{\$100 \times .07}{12} = \$0.583.$$

He must also subscribe for $100 of stock on which he may pay let us say $1 per month towards maturing $100 of stock with

which to extinguish his debt. Hence, the borrower of $100 would pay in all each month $1.583 per $100 borrowed.

The borrower often pays a smaller monthly instalment on stock than $1 per share of $100. Thus, he may pay on stock 50 cents per share of $100, or he may pay $1 per share on stock and for interest combined. Such smaller payments simply result in requiring a longer time to extinguish the debt than with payments of $1 per month. It is clear that with this arrangement, the stock of the borrower matures in just the same length of time that is required for the stock to mature for the man who is an investor only.

In essence, *the principle involved in such a building and loan operation is simply the sinking fund principle applied in a co-operative way to home building.*

To be sure, no separate fund is maintained as a sinking fund. The amounts required to amortize the debt of a borrower are given the advantages of investment in the association, and thus generally earn a higher rate of interest than could be obtained in a separate sinking fund.

75. Sources of profit. The largest item of profit is, of course, the interest paid by borrowers. There are, however, several minor sources of profit. The relative importance of these minor items varies in different associations. The following list will indicate the sources of profit to which it seems worth while to direct attention:

(1) Interest paid by borrowers.

(2) Gains from withdrawal of stock before it is matured. That is, the shareholder who withdraws his money before his stock is matured does not receive the full book value of the stock. He merely forfeits part of the earnings. Some associations pay as much as 6 per cent simple interest on what has been paid in, if payments were kept up for one year, but even with this rate of interest the transaction is likely to be a source of profit to an association operating on a 7 per cent basis.

(3) Profit on money borrowed at a lower interest rate than is received by the association.

An association is, in general, permitted by law to borrow amounts equal to a prescribed per cent of its assets, say 10 or 15 per cent. It often happens that an association can borrow at a lower rate than it realizes on its loans. This then clearly becomes a source of profit.

(4) Small entrance fees for admission to the association.

(5) Small fines for delinquencies in dues.

The last two items mentioned are of relatively little importance as sources of profit.

The expenses of conducting business are for secretarial work and clerical assistance. It is a fact that in many associations the expenses of doing business are remarkably small as compared to the amount of business done.

76. Distribution of profits to shareholders. In the distribution of profits to the shares, it is necessary to find the allotment of gain g per unit of book value (say per dollar of book value) in the hands of the association for the period for which dividends are to be found. The fact that the book value of instalment shares is not usually constant during the period makes it desirable to use some kind of an average value for the book value of each individual rather than the book value at the beginning of the period. The rules governing certain associations state explicitly that the dividend distribution shall be in proportion to the average amount of stock standing to the credit of each member during the dividend period.

The dividend period is so commonly six months that we shall use this period in our illustrative problems, although any other period might be used. In finding the average book value B_1 of the period for any individual shareholder, it is assumed that \$2 in the hands of the association for half the period, that is, for 3 months, obtains the same allotment of profit as \$1 in the hands of the association for six months. In general, \$$b$ in the hands of the association for $\frac{6}{b}$ months obtains the same allotment as \$1 for 6 months. The method of finding the aver-

age book value for an individual will be seen from illustrative problems.

If the average book values for the individual shareholders are $B_1, B_2, B_3, \ldots, B_n$. Then

$$B = B_1 + B_2 + \ldots + B_n$$

is the total book value.

Then the gain per unit of book value is

$$g = \frac{G}{B}, \tag{1}$$

where G is the net gain to be distributed.

PROBLEMS

1. The book value of 10 shares of stock in a building and loan association was \$379.20 at the beginning of a six months' period. The six payments of \$10 each per month in advance were made when due for the next six months' period. What is the average book value B_1 of the period that should be used in the distribution of profits to these 10 shares?

SOLUTION: The first \$10 of dues like the \$379.20 were invested 6 months.
The second \$10 of dues were invested 5 months.
The third \$10 of dues were invested 4 months.
. . . .
The sixth \$10 of dues were invested 1 month.

That is, \$10 was invested $6 + 5 + 4 + 3 + 2 + 1 = 21$ months.

Hence, $B_1 = \$379.20 + \frac{10 \times 21}{6} = \$414.20$.

2. A subscribed for 20 shares in a building and loan association between dividend dates so that he paid dues of \$20 each only four times. That is, he paid at the beginning of the third, fourth, fifth, and sixth months of the period.

SOLUTION: This is the same as having \$20 invested for $4 + 3 + 2 + 1 = 10$ months. But \$20 for 10 months is equivalent to $\frac{20 \times 10}{6} = \$33.33$ for 6 months.

Hence, $B_1 = \$33.33$.

3. The book value of 10 shares of stock was \$250 at the beginning of a six months' period. The shareholder was delinquent in dues the first three months of the period, so that he paid \$40 at the beginning of the fourth

month, and $10 each of the two succeeding months. What should be given as an average book value of his 10 shares? How much would his book value be if delinquency were completely covered by fines so that he could participate in the distribution of profits just as if he had not been delinquent?

4. The book value of 20 shares of stock in a building and loan association was $583.41 at the beginning of a six months' period. At the end of this period during which he paid dues of $20 per month it was found that the sum B of the average book values of holdings in the association was $126,178.36. The total net gain of the period was $4037.71. Find (*a*) the gain g per unit of book value, (*b*) the dividend allotted to the 20 shares mentioned above, (*c*) the book value of these 20 shares at the end of the six months' period, assuming that the dues were paid regularly. *Ans.* .032, $20.91, $674.32.

5. A stockholder pays $70 cash in advance for one share of paid-up stock in a building and loan association which matures when it accumulates to $100. The rates of profit for the first five semiannual periods were .032, .029, .030, .027, and .032 respectively. What is the book value of this share at the beginning of the sixth period?

6. A certain building and loan association issues shares at the beginning of each month. Just one month before the end of a six months' period, a man subscribed for 60 shares of stock on which he pays $60 per month. What average book value should be given to these 60 shares in the distribution of profits of the six months' period? What will be the dividend if the whole association showed a total net gain of $5262.50 and a book value $B = \$170,321.45$?

77. Shares issued in series. It is the practice of some building and loan associations to issue stock only at stated periods. The period between issues is usually 6 months, and the dates of issue generally coincide practically with dates of dividend additions to book values. All the shares issued at a given time are said to belong to the same series. Thus, they have Series A, Series B, and so on. Such a plan simplifies the problem of finding the average book values $B_1, B_2, \ldots, B_n$; for, except for delinquents, all have been paying the instalments of the entire six months' period.

PROBLEM: A has 20 shares of $100 each in each of two series of a given association. The one belongs to a series ending its first six months' period, and the other to a series ending its second six months' period. If A has

paid his dues of \$20 per month on each of these two sets of 20 shares, and the rate of profits per 6 months' period is .035, what are the dividends of the six months' period for the 20 shares of each series? *Ans.* \$2.45, \$6.74.

78. Withdrawal values. As stated in Art. 75, the shareholder who withdraws his money from a building and loan association does not forfeit all the earnings on his stock unless he withdraws soon after he subscribed for the stock. The withdrawal value is, however, not the full book value of the stock. There are obvious reasons why it would not be good business policy to make no charge for the trouble involved in handling withdrawals, and for the failure on the part of the withdrawing member to carry out his part in the plans of the association. The effect of withdrawals on the funds of the association can perhaps be grasped most readily by means of some concrete examples.

PROBLEMS

1. A man has paid \$40 per month on 40 shares of stock in a building and loan association for 66 payments. At the end of 66 months just before making the 67th payment, he withdraws his money for its withdrawal value which is the sum of his payments plus simple interest at 6 per cent per annum on his payments. If the value of the stock has been accumulating at 7 per cent per annum convertible monthly, what is the difference between book value and withdrawal value?

SOLUTION: Book value = \$40 $\cdot\ s_{\overline{66}|}$ at rate $7/12\%$,

$$= \$40\ \frac{(1.005833)^{66} - 1}{.005833333}\ (1.005833) = \$3227.62.$$

To find the simple interest, we note that the first payment was invested 66 months, the second 65, the third 64, etc., and the last 1 month. This is equivalent to \$40 invested for $66 + 65 + 64 + \ldots + 1 = 2211$ months = 184.25 years (see Art. 149). The simple interest on \$40 for 184.25 years at 6 per cent = \$442.20. Total withdrawal value = $(66 \times 40) + 442.20$ = \$3082.20.

Hence, book value less withdrawal value = \$3227.62 — \$3082.20 = \$145.42.

2. Given the same conditions as in problem 1 except that interest paid on withdrawals is to be 5 per cent simple interest. What is the difference between book value and withdrawal value?

3. Given same conditions as in problem 1 except the value of this stock has been accumulating at 6 per cent convertible monthly instead of 7 per cent. What is the difference between book value and withdrawal value? *Ans.* \$67.66.

4. In a certain building and loan association, stock matures to \$200 a share. A stockholder has paid \$10 per month into the association for 96 payments. Just before making the 97th payment he asks the withdrawal value. The rules of the association make the withdrawal value the sum of the payments plus simple interest at 4 per cent per annum on his payments. Stock has been accumulating at the rate of 6½ per cent per annum convertible monthly. What is the difference between the book and withdrawal values?

5. A certain association matures stock to \$200 a share. A share of paid-up stock was purchased for \$140. The value of the stock accumulates at a rate such as to mature the stock in four years. The withdrawal value in this association is the original value paid for the stock plus interest at 5 per cent compounded annually. What is the difference at the end of three years from the date of buying between the book value and the withdrawal value of the share? *Ans.* \$20.87.

79. Time required for stock to mature, rate of interest given. Let F be the face of the stock that is being paid for by monthly payments M. The problem proposed is to find the time n required for the stock to mature at an effective rate of interest i. This problem is simply that of finding the time for an annuity certain of annual rent $12M$ in 12 monthly instalments to accumulate to an amount F (see Chapter II, Art. 33).

It is of importance to examine the various cases which arise depending on the situation at the time of maturity.

Thus, if the stock matures immediately after making a monthly payment, we may write the equation

$$M(1+i)^n + M(1+i)^{n-\frac{1}{12}} + M(1+i)^{n-\frac{2}{12}} + \cdots + M(1+i)^{\frac{1}{12}} + M = F, \qquad (2)$$

or $$(1+i)^n + (1+i)^{n-\frac{1}{12}} + (1+i)^{n-\frac{2}{12}} + \cdots + (1+i)^{\frac{1}{12}} = \frac{F}{M} - 1. \qquad (3)$$

By summing the geometrical progression in the left hand member, we have (see Art. 152)

$$\frac{(1+i)^{n+\frac{1}{12}}-(1+i)^{\frac{1}{12}}}{(1+i)^{\frac{1}{12}}-1}=\frac{F}{M}-1, \tag{4}$$

or

$$\frac{(1+i)^{n+\frac{1}{12}}-1}{(1+i)^{\frac{1}{12}}-1}=\frac{F}{M}. \tag{5}$$

If the stock matures just before making a monthly payment, we may write in place of (2) the equation

$$M(1+i)^n+M(1+i)^{n-\frac{1}{12}}+M(1+i)^{n-\frac{2}{12}}+\ldots+M(1+i)^{\frac{1}{12}}=F, \tag{6}$$

and in place of (4), we obtain

$$\frac{(1+i)^{n+\frac{1}{12}}-(1+i)^{\frac{1}{12}}}{(1+i)^{\frac{1}{12}}-1}=\frac{F}{M}. \tag{7}$$

If the stock matures by taking only part of the last monthly payment, say by taking fM, where f is a proper fraction, we obtain in place of (4)

$$\frac{(1+i)^{n+\frac{1}{12}}-(1+i)^{\frac{1}{12}}}{(1+i)^{\frac{1}{12}}-1}=\frac{F}{M}-f. \tag{8}$$

If the stock matures between two monthly payments, let t be the time in years equivalent to the largest number of integral months contained in the time n. Then we write in place of (2)

$$M(1+i)^n+M(1+i)^{n-\frac{1}{12}}+M(1+i)^{n-\frac{2}{12}}+\ldots+M(1+i)^{n-t}=F, \tag{9}$$

or, $$(1+i)^n+(1+i)^{n-\frac{1}{12}}+(1+i)^{n-\frac{2}{12}}+\ldots+(1+i)^{n-t}=\frac{F}{M}. \tag{10}$$

Then in place of (4), we have

$$\frac{(1+i)^{n+\frac{1}{12}}-(1+i)^{n-t}}{(1+i)^{\frac{1}{12}}-1}=\frac{F}{M}. \tag{11}$$

By comparing equations (4), (5), (7), (8) and (11), it is easily shown, remembering that $n - t < \frac{1}{12}$, that for given values of F, M, and i, the value of n that satisfies (4) and (5) is smaller than values which satisfy (7), (8), or (11).

Since f and t are unknown, it is impractical to use equations (8) or (11) to solve for n in any given problem. We may, however, find an approximate value from (4) or (5). Assume that $n = a$ is such an approximate value. We may then find the accumulated value of the monthly annuity

$$S = (1 + i)^t + (1 + i)^{t - \frac{1}{12}} + (1 + i)^{t - \frac{2}{12}} + \ldots + 1, \tag{12}$$

where t is the time in years equal to the largest number of integral months in a.

By summing the progression in (12), we have

$$S = \frac{(1 + i)^{t + \frac{1}{12}} - 1}{(1 + i)^{\frac{1}{12}} - 1}. \tag{13}$$

Then with S known, we make the adjustments as to further time required and as to payment of part of the dues of another month. The method of making such an adjustment will be clear from illustrative exercises. When a has a value just slightly less than an integral number of months, m may well be taken as the next integral number of months larger than a. (See problem 3 below.)

Before proceeding to numerical exercises, we may give the general solution of (4) or (5) for n. Thus, from (5), we have

$$(1 + i)^{n + \frac{1}{12}} = \frac{F}{M}[(1 + i)^{\frac{1}{12}} - 1] + 1. \tag{14}$$

Then

$$\left(n + \frac{1}{12}\right) \log (1 + i) = \log \left\{\frac{F}{M}[(1 + i)^{\frac{1}{12}} - 1] + 1\right\}, \tag{15}$$

and

$$n = -\frac{1}{12} + \frac{\log \left\{\frac{F}{M}[(1 + i)^{\frac{1}{12}} - 1] + 1\right\}}{\log (1 + i)}. \tag{16}$$

If the rate of interest quoted is a nominal rate j convertible m times a year, (16) may be written as

$$n = -\frac{1}{12} + \frac{\log\left\{\frac{F}{M}\left[\left(1+\frac{j}{m}\right)^{\frac{m}{12}} - 1\right] + 1\right\}}{m \log\left(1+\frac{j}{m}\right)}. \tag{17}$$

It should be remembered that this is an exact formula for the time required only when the stock matures immediately after a monthly payment. In other cases, this value of n is an approximation. We shall show the plans of adjustment in illustrative problems for finding closer approximations from the first approximation.

PROBLEMS

1. If a building and loan association yields for the investor a nominal rate of 7 per cent convertible monthly, find the time required for payments of $1 per month to mature to $100.

Solution: In this case, $j = .07$, $m = 12$.

From (17), we have for the approximate time,

$$n = -\frac{1}{12} + \frac{\log\left[100\left(1.00\tfrac{7}{12} - 1\right) + 1\right]}{12 \log\left(1.00\tfrac{7}{12}\right)} = -\frac{1}{12} + \frac{\log 19 - \log 12}{12 \log\left(1.00\tfrac{7}{12}\right)}$$

$$= 6.501 = 6 \text{ years, } 6 \text{ months, } 0 \text{ days.}$$

Hence, the stock matures in 6 years, 6 months just after making the 79th monthly payment.

Check by finding $S = \dfrac{\left(1.00\tfrac{7}{12}\right)^{79} - 1}{.00\tfrac{7}{12}}$ as given by formula (1) Art. 32.

2. Given the same conditions as in problem 1 except that the interest is at the rate of 6 per cent instead of 7 per cent. Find the time of maturity.

Solution: In this case, from (17)

$$n = -\frac{1}{12} + \frac{\log 1.5}{12 \log 1.005},$$

$$= 6.6912 \text{ years,}$$

$$= 6 \text{ years, } 8.294 \text{ months.}$$

The accumulated value at the end of 6 years, 8 months after making the 81st payment is

$$S = \frac{(1.005)^{81} - 1}{.005},$$

$$= 99.5602.$$

If x is the time in fractions of a month required to mature the stock,

$$(99.5602)\ (1.005)^x = 100,$$

$$x = \frac{\log 100 - \log 99.5602}{\log 1.005} = 0.8837 \text{ month},$$

$$= 27 \text{ days}.$$

But for such a short period as part of a month, we may use simple interest. Then

$$(.005)\ (99.56)\ x = 100 - 99.5602,$$

$$.4978\, x = 0.4398,$$

$$x = .8835 \text{ month} = 27 \text{ days}.$$

Hence, the stock matures in 6 years, 8 months and 27 days.

3. Given payments of 50 cents per month on a share of $100 in a building and loan association. Assume that this yields for the investor a nominal rate of 6 per cent convertible monthly. Find the time required for payments to mature.

SOLUTION: From (17),

$$n = -\frac{1}{12} + \frac{\log 2}{12 \log 1.005},$$

$$= 11.4978 \text{ years}.$$

This is so near to 11.5 years that we may well find the accumulated value to the end of 11.5 years just after a payment $0.50 at that time. This value is

$$S = \$0.50\, \frac{(1.005)^{139} - 1}{.005},$$

$$= \$100.03.$$

Hence, we should take $0.50 — $0.03 = $0.47 of the 139th payment instead of $0.50 to make the stock mature.

4. Given $0.50 per month payments on a share of $100 in a building and loan association. If the returns to the investor give an effective rate of 7 per cent convertible annually, when will the stock mature?

80. The effective rate of interest on money invested in instalment shares. Given the time n in which monthly instalments of M per month invested in a building and loan association accumulate to the face F, it is required to find the effective rate of interest i which this yields the investor. This is simply an illustration of the rate earned on an annuity. (See Art. 36.)

As indicated in Art. 79, the stock may mature at the time of a monthly payment, or at a time between monthly payments. If it matures at the time of a monthly payment, we have

$$M(1+i)^n + M(1+i)^{n-\frac{1}{12}} + M(1+i)^{n-\frac{2}{12}} + \dots + M(1+i)^{\frac{1}{12}} + fM = F, \quad (18)$$

where $$0 \overline{\overline{<}} f \overline{\overline{<}} 1.$$

From (18), $$M\frac{(1+i)^{n+\frac{1}{12}} - (1+i)^{\frac{1}{12}}}{(1+i)^{\frac{1}{12}} - 1} = F - fM. \quad (19)$$

If $f = 0$, the stock matures at the end of a month without taking any part of the monthly payment at maturity date. If $f = 1$, the stock matures just after taking a full monthly payment at maturity date. If the stock matures between monthly payments, we have in place of (18) the equation

$$\left\{M(1+i)^t + M(1+i)^{t-\frac{1}{12}} + \dots + 1\right\}(1+i)^{\frac{f'}{12}} = F,$$

where t is defined as in (9), and f' is the fraction of a month from the last payment date to maturity. This will perhaps be best understood by concrete illustrations.

PROBLEMS

1. The Home Building and Loan Association of Urbana, Illinois, issued stock August 15, 1912, maturing to \$100 per share with monthly payments of \$1 per share. This stock matured Feb. 15, 1919, by accepting \$0.40, on the 79th payment of \$1. What is the effective rate of interest earned?

SOLUTION: From (19), we have the equation

$$\frac{(1+i)^{\frac{79}{12}} - (1+i)^{\frac{1}{12}}}{(1+i)^{\frac{1}{12}} - 1} = \$99.60, \quad (A)$$

or $$(1+i)^{\frac{79}{12}} - 100.6\,(1+i)^{\frac{1}{12}} + 99.6 = 0. \tag{B}$$

to be solved for i.

This equation may be solved by successive approximations. We know from the result of problem 1, Art. 79, that i is greater than the effective rate that corresponds to a nominal rate of .07 convertible monthly. Hence,

$$i > (1.00\tfrac{7}{12})^{12} - 1 > 0.072.$$

Hence, if we let

$$i = .072 + h \text{ in (B)},$$

we have a small number h to be determined.

Substitution in (B) gives

$$(1.072+h)^{\frac{79}{12}} - 100.6\,(1.072+h)^{\frac{1}{12}} + 99.6 = 0. \tag{C}$$

Expand the binomials but retain only the first degree terms in h. This gives

$$(1.072)^{\frac{79}{12}} + \frac{79}{12}(1.072)^{\frac{67}{12}}h - 100.6\,(1.072)^{\frac{1}{12}} - \frac{100.6}{12}(1.072)^{-\frac{11}{12}}h + 99.6 = 0, \tag{D}$$

$$\text{or } 1.58046 + 9.70584\,h - 101.1845^{*} - 7.86571\,h + 99.6 = 0,$$

$$\text{or } 1.8401\,h - .0040 = 0,$$

$$h = .0022.$$

Hence, $i = .0742$.

Check the accuracy of $i = .0742$ by substitution in

$$\frac{(1+i)^{\frac{79}{12}} - (1+i)^{\frac{1}{12}}}{(1+i)^{\frac{1}{12}} - 1} = 99.60.$$

If greater accuracy is desired, make

$$i = .0742 + h,$$

and proceed as above to a closer approximation.

2. If in problem 1, the stock had matured by accepting the full amount of the 80th payment, what would have been the effective rate earned? *Ans.* .0706.

3. A building and loan association matures stock in shares of \$100 on which the payments were \$.50 per month, at the end of 11 years and 6

* This term should be calculated with seven place tables.

months without accepting any part of the 139th payment; what effective rate of interest is earned?

4. If in problem 1, the stock had matured at the end of 78.5 months, what would have been the effective rate earned?

81. The rate of interest from the standpoint of the borrower. If the borrower should consider the interest on his loan and the dues for the purchase of stock to be entirely separate transactions, the rate of interest the borrower pays is simply that specified in the loan. But these two features are not separate in practice because the borrower is an investor in stock. In order to extinguish his debt, the borrower has the advantage of creating a sinking fund at the rate of interest earned on stock of the association.

Hence, the question arises as to the rate a borrower can afford to pay in a building and loan association in place of a smaller rate when the money to pay off the principal has to be invested in ordinary savings accounts or in some form of sinking fund instead of earning the rates on stock in a building and loan association. Some concrete problems will afford interesting comparisons.

PROBLEMS

1. A man building a house can borrow from a building and loan association on a 7 per cent nominal interest basis in which the stock on which \$1 per month is paid would mature to \$100 in 78 months just after making the 79th monthly payment. He can also borrow from another source at 6 per cent interest payable monthly in advance, and invest the balance of what he would put into the building and loan association into a sinking fund at 4 per cent, interest payable monthly. How much would he have saved at the end of the 78 months by choosing the building and loan proposition?

Solution: On each \$100 borrowed, he would pay each month to the building and loan association \$1.5833. Under the second proposition, his interest payments per month would be \$0.50. Hence, with the payment of \$1.5833 per month, he would under the second plan have available each month

$$\$1.0833$$

to put into the sinking fund at 4 per cent convertible monthly. This would at the end of 78 months just after making the 79th payment amount to

$$S = 1.0833 \frac{(1.00\frac{1}{3})^{79} - 1}{.00\frac{1}{3}} = \$97.72.$$

He would still owe \$100 — \$97.72 = \$2.28 per \$100 borrowed when his debt would have been discharged by taking the building and loan association proposition.

2. Same as problem 1 except that the balance of \$1.0833 per month was put into a savings bank at 4 per cent convertible semiannually.

Solution: At the end of the 78 months,

the first	\$1.0833 would amount to	$(1.0833)\ (1.02)^{13}$,
the second	1.0833 would amount to	$(1.0833)\ (1.02)^{12^5/_6}$,
the third	1.0833 would amount to	$(1.0833)\ (1.02)^{12^2/_3}$,
. . . .	. . . .	. . . .
the 78th	1.0833 would amount to	$(1.0833)\ (1.02)^{1/_6}$,
the 79th	1.0883 would be simply	1.0833.

Adding together these accumulations, we have

$$(1.0833)\ [1 + (1.02)^{1/_6} + (1.02)^{1/_3} + \ldots + (1.02)^{12^2/_3} + (1.02)^{12^5/_6} + (1.02)^{13}]$$

Summing the geometrical progression in brackets, we have

$$1.08333 \cdot \frac{(1.02)^{13^1/_6} - 1}{(1.02)^{1/_6} - 1} = 1.08333 \frac{1.297883 - 1}{1.003306 - 1} = \$97.61.$$

The borrower would, under the savings bank plan, still owe \$100 — \$97.61 = \$2.39 per \$100 borrowed when his debt would have been discharged by taking the building and loan association proposition.

3. Given the same conditions as in problem 1, except that the interest on the sinking fund is at 4½ per cent convertible monthly.

4. Given same conditions as in problem 2 except that the savings bank rate is 4½ per cent payable semiannually. Compare the two propositions.

We next propose to treat the problem of finding the effective rate of interest paid by the borrowing shareholder when the interest and dues are together considered as a single sum for the payment of interest and principal. For this purpose, let

F be the amount borrowed,
g the nominal rate of interest,
M the monthly dues paid on stock,
n the number of years required for stock to mature,

i the unknown effective rate of interest paid by the borrower. Then, we have

$$\frac{Fg}{12} + M + 12\left(\frac{Fg}{12} + M\right)a_{\overline{n}|}^{(12)} = F, \tag{20}$$

if the stock matures just after making a full monthly payment. From (20), we obtain

$$a_{\overline{n}|}^{(12)} = \frac{12\,F - Fg - 12\,M}{12\,(Fg + 12\,M)}.$$

Replacing $a_{\overline{n}|}^{(12)}$ by its value, Art. 35, we have

$$\frac{1 - (1 + i)^{-n}}{12\,[(1 + i)^{\frac{1}{12}} - 1]} = \frac{12\,F - Fg - 12\,M}{12\,(Fg + 12\,M)}. \tag{21}$$

We solve this equation for i by successive approximations in a manner similar to that used in Art. 80.

If the last payment required is not a full monthly payment or if stock matures between monthly payments, the equation (20) requires slight modifications similar to those given in Art. 80.

Concrete illustrative problems will perhaps make the solution clearer than any further formal discussion.

In the solution of (21) a seven-place logarithmic table is usually desirable.

PROBLEMS

1. A borrowing shareholder in a building and loan association operating on a 6 per cent nominal interest rate convertible monthly pays \$5 of interest and \$5 of dues per month on \$1000 which he has borrowed. If his stock matures at the end of 11 years and 3 months just after making the monthly payment of interest and dues required at that time, what effective rate of interest has he paid on his loan when interest and dues together are regarded as a single sum for the payment of interest and principal?

Solution: From (21) above

$$\frac{1 - (1 + i)^{-11^1/_4}}{12\,[(1 + i)^{1/_{12}} - 1]} = \frac{12000 - 60 - 60}{12\,(60 + 60)} = 8.25,$$

$$\text{or } 1 - (1 + i)^{-11^1/_4} = 99\,[(1 + i)^{1/_{12}} - 1],$$

$$\text{or } 1 - (1 + i)^{-11^1/_4} - 99\,(1 + i)^{1/_{12}} + 99 = 0. \tag{A}$$

By considering the results of problem 3, Art. 79, it seems that the rate would probably not differ much from .06. Hence, assume

$$i = .06 + h \text{ and substitute in (A).}$$

This gives

$$1 - (1.06 + h)^{-11^1/_4} - 99\,(1.06 + h)^{1/_{12}} + 99 = 0.$$

Expanding the binomials and retaining only 1st degree terms in h, we have

$$1-(1.06)^{-11^1/_4} + 11\tfrac{1}{4}\,(1.06)^{-12^1/_4}\,h - 99\,(1.06)^{1/_{12}} - \frac{99}{12}\,(1.06)^{-11/_{12}}\,h + 99 = 0,$$

$$1 - 0.51917 + 5.51005\,h - 99.48189 - 7.82090\,h + 99 = 0,$$

$$-2.31085\,h - .00106 = 0,$$

$$h = -0.00046.$$

Hence, $i = 0.05954$.

2. In the Commercial Building and Loan Association of Urbana, Illinois, operating on a 7 per cent basis, a borrowing shareholder pays \$7 in interest and \$12 in dues per month on \$1200. In May, 1919, such stock matured in 78 months by taking interest for the month but only \$2.40 of the 79th payment of dues for stock. What effective rate of interest was paid by the borrowing shareholder when interest and dues together are considered as a single sum for repayment of interest and principal?

MISCELLANEOUS PROBLEMS

1. A man paid \$20 per month in dues on stock of \$2000 to the Commercial Building and Loan Association of Urbana, Illinois, for 78 months but paid only \$4 of the 79th payment, when his stock was declared mature. What effective rate of interest did his money earn?

2. The successive semiannual distributions of profits in a certain building and loan association are at an average rate of .0324. What is the approximate effective rate of interest? With monthly payments of \$1 per month per share, about how long will it take for the stock to mature when issued in shares of \$100 each?

3. A building and loan association has \$495,320 loaned to its members on a 7 per cent basis, of which \$45,000 is borrowed at a bank at the rate of 6 per cent payable semiannually. The expenses of doing business for a six months' period amount to \$725. Profits from withdrawals amount to \$462. Fees and fines amount to \$92. The average book value of the stock for the six months' period is \$452,235. Assuming that all interest due is paid, what is the rate of profit for the six months' period?

4. A subscriber to building and loan stock on which he is paying \$20 per month on \$2000 of stock, needs his money at the end of the 66th month just before making the 67th payment. He can withdraw his money by receiving what he has paid in with simple interest at 6 per cent per annum. If this stock on which he is paying would mature in 78 months just after making the 79th payment, what is the difference between the book value and the withdrawal value of his stock? Assuming that he could borrow money at 8 per cent simple interest to pay his remaining dues to mature his stock, find the rate at which he could afford to borrow an amount equal to the withdrawal value until his stock matures, instead of withdrawing his money. *Ans.* $\begin{cases}\text{Difference \$72.71.} \\ \text{At rate} \quad 12.2 \text{ per cent.}\end{cases}$

HINT: It will save time in the solution to accept from problem 1, Art. 79, the fact that stock maturing as specified in this problem is earning 7 per cent per annum convertible monthly.

5. Some building and loan associations have the following rule for withdrawals: "Withdrawing members shall be entitled to receive the full amount of dues paid in on the shares sought to be withdrawn and simple interest thereon at the rate of 3 per cent per annum to the date of withdrawal, and in addition thereto such a proportion of the profits apportioned to said shares as the board of directors may determine."

Solve problem 4 under this rule if the directors had determined that proper apportionment was $1/3$, and payments in the hands of the association had accumulated at the rate of 7 per cent convertible monthly. *Ans.* \$78.36; 13½ per cent.

6. Given the same conditions as in problem 4, except that the interest payable on withdrawal is simple interest at 5 per cent per annum, solve the problem.

7. What rate of simple interest has been earned on building and loan stock at the date of maturity, when a man has paid \$1 per month on a share of stock that matured to \$100 at the end of 78 months just after making the 79th payment? If he had paid \$.50 per month on a share of \$100 to the same association earning 7 per cent convertible monthly, what would his simple interest rate be at the time of maturity of his stock? Should you regard the second plan as a better financial proposition for the investor, because his simple interest rate is higher? Give reasons for your answer.

8. In certain sections of the country, there are building and loan associations operating on the basis of a 10 per cent interest rate convertible monthly. If such an association should be able to accumulate money paid in at the rate of 10 per cent convertible monthly, what time would be required for payments of \$1 per month to cause a \$100 share to mature?

CHAPTER VII

THEORY OF PROBABILITY WITH SPECIAL REFERENCE TO ITS APPLICATION IN INSURANCE

82. Meaning of probability. We propose in this chapter to study the relative frequency of the occurrence of future events when an opportunity exists for the events to happen. For example, an urn contains five white and seven black balls. One ball is drawn at random, and then replaced. This process is continued indefinitely. What proportion of the balls drawn will be white?

The event is said to happen if a white ball is drawn, and to fail if a black ball is drawn. The event may happen by drawing any one of five balls. That is, the number of ways of happening is five. The number of ways in which the event may happen or fail is twelve. It is reasonable then to expect that, in the long run, $^5/_{12}$ of the balls drawn will be white balls. We express this result in another way by saying that the probability or chance of drawing a white ball at a single drawing is $^5/_{12}$. This illustrates the following **definition of probability**:—

If all the happenings and failings of an event can be analyzed into $r + s$ possible ways each of which is equally likely, and if the event will happen in r of these ways, and fail in s of them, the probability that the event will happen is $\frac{r}{r+s}$ and the probability that it will fail is $\frac{s}{r+s}$.

In applying this definition of probability, the fact must not be overlooked that the ways are assumed to be equally likely. Here is the main difficulty in the application of the definition.

EXAMPLE: What is the probability that a man in good health aged 20, will die before he is of age 21? We might hold that the event can happen

in only one way and fail in only one way, and that ½ is, therefore, the probability that he will die within the specified time. What is the flaw in this view?

The French mathematician and philosopher, D'Alambert, said "There are two possible cases with respect to every event, one that it will occur, the other that it will not occur." Hence, the chance of every event is ½ and the definition of probability is meaningless. Criticise this view.

The expression "equally likely" indicates that we have no reason to expect the event to happen more frequently in one assigned way than in another assigned way.

From the definition of probability it follows directly *that the sum* $\frac{r}{r+s} + \frac{s}{r+s}$ *of the probability that an event will happen and that it will fail is 1, the symbol for certainty.*

Many ideas about probability had their origin in games of chance, but these ideas have grown to be very important, mainly because of their application in insurance, in the mathematics of statistics, and in many branches of science.

83. Probability derived from observation. There are important classes of events, such as those against which we insure, in which it is impossible to enumerate all the equally likely ways in which the event can happen or fail. Still it is possible to make a practical determination of what is called the probability of the event by observation.

If it be observed that an event has happened m times in n possible cases (n a large number), then, in the absence of further knowledge, it may be assumed for many practical purposes that $\frac{m}{n}$ is the best estimate of the probability of the event, and that confidence in this estimate increases as n increases.

Such determinations of probability are of much practical value in statistics and insurance. For example, according to the American Experience Table of Mortality, of 78,106 men living at age 40, the number living ten years later will be 69,804. The

probability that a man aged 40 will live ten years is taken to be $\frac{69{,}804}{78{,}106} = .8937.$

The fraction $\frac{m}{n}$ defined above is the **relative frequency** of the happenings of the event. The precise conception of the probability of the event may be regarded as the limiting value of $\frac{m}{n}$ when n increases without bound. It is necessary to assume the existence of such a limit in making exact statements of the theory of probability. In statistical problems, the limit of $\frac{m}{n}$ cannot, in general, be determined, but we can find approximations to the limit which are satisfactory for many practical purposes.

84. Expectation of money. When p is the probability that a person will win an amount of value m, the **expectation** of the person is defined as pm. That is to say, we use the probability p, times the sum of money m to indicate the strength of our expectation of winning the money in a single trial.

PROBLEMS

1. An urn contains 10 white and 15 black balls. What is the probability that a ball drawn at random will be white?

2. An urn contains 5 times as many white balls as black balls and one ball is drawn out at random. What is the probability that the ball drawn is white?

3. According to a mortality table, it appears that of 100,000 persons of age 10, the number that reach the age of 65 is 49,341. What is the probability that a child aged 10 will live to be 65?

4. According to measurements by Karl Pearson on the stature of 1078 mature men of a certain class it was found that the number of statures between 66.5 inches and 67.5 inches was 148. Give from these figures the best estimate of the probability that the stature of a man taken at random falls between 66.5 and 67.5 inches.

5. According to statistics of births in France in a certain year, out of 787,446 children born, there were 398,909 boys and 388,537 girls. Give

from these figures the best estimate of the probability that a child to be born will be a boy?

6. A gambler is to win \$60 if he throws an ace in a single throw with a die. What is the value of his expectation?

85. Number of ways of doing two or more things together. It is necessary in establishing certain elementary principles of probability that we know the number of ways of doing two things together when we have given the number of ways of doing each of them separately. For example, two positions are to be filled in an office, the one is that of stenographer and the other that of office boy. There are ten applicants for the position of stenographer, and five for that of office boy. In how many ways can the two positions be filled?

There are ten ways of filling the position of stenographer, and with each of these, there is a choice of five office boys. Hence the two positions can be filled in $10 \times 5 = 50$ ways. This example illustrates the following

FUNDAMENTAL PRINCIPLE. *If one thing can be done in m different ways, and if after it is done in one of these ways another thing can be done in n ways, then the two together can be done in the order stated in mn ways.*

For, corresponding to each of the m ways of doing the first thing, there are n ways of doing the second thing. That is to say, there are n ways of doing the two together for each way of doing the first thing. Hence, there are in all mn ways of doing the two together. The principle can be extended in an obvious manner to find the number of ways of doing three or more things together.

86. Meaning of permutations of things all different. By a **permutation** of a set of things we mean an arrangement of all or part of the set of things. By the expression, "number of permutations of n things taken r at a time," is meant the number of permutations consisting of r things which can be formed from n different things. Thus, the permutations of the letters *abc* taken two at a time are—*ab, ba, ac, ca, bc, cb.*

87. Number of permutations of things all different. The symbol $_nP_r$ is used to represent the number of permutations of n things taken r at a time.

This number of permutations of n things taken r at a time is found as follows: The number $_nP_r$ is the same as the number of ways of filling r different places with n different things. We may represent the n things by $a_1, a_2, \ldots, a_n$ and ask for the number of permutations of r letters that can be formed from them. For the first place, there is a choice of n letters, for the second a choice of $n - 1$ letters, for the third a choice of $n - 2$ letters, and so on. For the rth place there is a choice of $n - r + 1$ letters. Hence, from the fundamental principle (Art. 85), we have

$$\boldsymbol{{}_nP_r = n\,(n - 1) \ldots (n - r + 1).} \tag{1}$$

When $r = n$, (1) becomes

$$_nP_n = n(n - 1) \ldots 2 \cdot 1 = n!\,^* \tag{2}$$

EXERCISES AND PROBLEMS

1. A man has three suits of clothes, and five neckties. In how many ways can he dress by changing suits and neckties?

2. A man has three suits of clothes, five neckties and three hats. In how many ways can he dress by changing suits, neckties and hats?

3. Five persons enter a railway coach in which there are seven vacant seats. In how many ways can they take their places? *Ans.* 2520.

4. Write all the permutations of the letters *abcd* when taken (1) three at a time, (2) all at a time.

5. How many permutations of the letters of the word Texas?

88. Meaning of a combination. A **combination** is a set of things or elements without reference to the order of individuals within the set. Thus, *ab* and *ba* are the same combination.

By the number of combinations of n things taken r at a time is meant the number of combinations of r things that can be

* The symbol $n!$ is an abbreviation for $n(n - 1) \ldots 2 \cdot 1$, and is called "factorial n."

formed from n things. Thus the combinations of *abcd* taken three at a time are, *abc*, *abd*, *acd*, *bcd*.

89. Number of combinations of things all different. Let ${}_nC_r$ stand for the number of combinations of n things taken r at a time. The formula for ${}_nC_r$ is found by finding the relation of ${}_nC_r$ to ${}_nP_r$.

Take any one combination of r things. With this combination, $r!$ permutations can be made. Take a second combination. With this combination $r!$ permutations can be made. There are thus $r!$ permutations for each combination. Hence,

$$ {}_nC_r\, r! = {}_nP_r, $$

whence

$$ {}_nC_r = \frac{{}_nP_r}{r!}. $$

Since

$$ {}_nP_r = n(n-1)\ldots(n-r+1), \qquad \text{(Art. 87)} $$

we have

$$ {}_nC_r = \frac{n(n-1)\ldots(n-r+1)}{r!}. \tag{3} $$

Multiply numerator and denominator by $(n-r)!$.

This gives

$$ {}_nC_r = \frac{n!}{r!\,(n-r)!}. \tag{4} $$

90. Binomial coefficients equal ${}_n\mathbf{C}_r$ (r = 1 to n). For our use in repeated trials in probability, it is important to observe that ${}_nC_r$ is the coefficient of the $(r+1)$th term of the binomial expansion

$$ (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots + nax^{n-1} + x^n. $$

The binomial theorem for positive integral exponents may, therefore, be written in the form

$$ (a+x)^n = a^n + {}_nC_1\, a^{n-1}x + {}_nC_2\, a^{n-2}x^2 + \ldots + {}_nC_n x^n. $$

EXERCISES AND PROBLEMS

1. How many different committees of 4 each can be selected from 12 men? *Ans.* 495.

2. A woman with 9 friends to invite can have how many dinner parties with four guests without having the same company of four twice? *Ans.* 126.

3. Prove ${}_nC_r = {}_nC_{n-r}$.

4. How many different assemblages of 500 persons can be selected from an aggregate of 502 persons? *Ans.* 125,751.

5. A committee of 5 is to be chosen from 6 Englishmen and 3 Americans. If the committee is to contain at least two Americans, in how many ways may the committee be chosen?

6. Six coins are tossed. What is the probability that exactly two of them are heads?

Solution: Since each coin can face in two ways, the 6 can face in 2^6 ways or 64 ways. The two coins can be selected in ${}_6C_2 = 15$ ways. Hence the probability is 15/64.

7. From an urn containing 6 black balls and 5 white balls, 4 are drawn at random. Find the probability that two are white and two are black? *Ans.* $\frac{5}{11}$.

8. A bag contains 5 white, 8 black and 6 red balls. If 3 balls are drawn at random, what is the probability that (1) all are black, (2) 2 black, and 1 red, (3) 1 white, 1 black, and 1 red?

9. From 4 sophomores, 5 juniors, and 5 seniors, a committee of 3 is to be selected by lot. Find the probability that it will consist (1) of 1 sophomore, 1 junior and 1 senior, (2) of three seniors, (3) of 1 sophomore and 2 juniors? *Ans.* (1) $\frac{25}{91}$, (2) $\frac{5}{182}$, (3) $\frac{10}{91}$.

10. What is the probability of throwing a total of 7 in a single throw of two dice?

91. Compound events. When we consider the probability of the joint occurrence of two or more events it is desirable to distinguish between **independent** and **dependent** events. The events of a set are said to be independent or dependent according as the occurrence of any one of them does not or does affect the occurrence of the others.

Independent Events. Theorem I. *The probability that all of a set of independent events will happen on a given occasion*

when all of them are in question is the product of their separate probabilities.

Let $p_1, p_2, \ldots, p_r$ be the separate probabilities of the events. Out of a large number, n, of trials, the first will happen on p_1n occasions. Out of these, the second will happen on $p_2(p_1n)$ occasions. Continuing this process, all the r events should happen on $p_1p_2 \ldots p_rn$ occasions. The probability p of the joint occurrence is then

$$p = \frac{p_1p_2 \ldots p_rn}{n} = p_1p_2 \ldots p_r, \tag{5}$$

and the theorem is established.

DEPENDENT EVENTS. THEOREM II. *If the probability of a first event is p_1, and if, after this has happened, the probability of a second is p_2, the probability that both events will happen in the specified order is p_1p_2.*

The extension to any number of events is obvious.

The probability of the joint occurrence of two events is sometimes called the probability of "both and," because the problems are often phrased to ask for the probability of both of the two events.

EXERCISE. The probability that A will live five years is $^9/_{10}$ and that B will live five years is $^7/_{10}$. What is the probability that both A and B will live five years?

92. Mutually exclusive events. Events are said to be **mutually exclusive** when the occurrence of any one of them on a particular occasion excludes the occurrence of any other on that occasion. The probability in such a situation is sometimes called the probability of "either or," because the problems are often phrased so as to ask for the probability of the occurrence of either one event or another.

THEOREM. *If the separate probabilities of r mutually exclusive events are respectively $p_1, p_2, \ldots p_r$, the probability that one of these events will happen on a particular occasion when all of them are in question is the sum $p_1 + p_2 + \ldots + p_r$.*

This proposition follows at once from the definition of mutu-

ally exclusive events. Thus, the probability of throwing either an ace or a deuce in a single throw of a die is

$$1/6 + 1/6 = 1/3.$$

PROBLEMS

1. What is the chance of throwing either a 4, 5, or 6 in a single throw with a die?

2. What is the probability of throwing 3 aces in a throw of 3 dice? *Ans.* $\frac{1}{216}$.

3. The probability that A will live 20 years is $3/5$ and the probability that B will live 20 years is $1/3$. What is the probability that both will live 20 years?

4. A traveler has four railroad connections to make. If the probability is $4/5$ that he would make any particular connection taken alone, what is the probability that he will make all four connections? *Ans.* $\frac{256}{625}$.

5. A bag contains 3 white, 4 black, and 5 red balls. If a ball is drawn from the bag at random, what is the probability that it will be either white or red?

6. From a frequency distribution prepared by Karl Pearson, the probability that the stature of a mature man taken at random from a certain class of men will fall between 5 feet 7.5 inches and 5 feet 9.5 inches is $323/1078$. The probability that it will fall between 5 feet 9.5 inches and 5 feet 10.5 inches is $128/1078$. What is the probability that it will fall between 5 feet 7.5 inches and 5 feet 10.5 inches?

93. Repeated trials. THEOREM. *If p is the probability that an event will happen in a single trial, the probability that it will happen exactly r times in n trials is equal to*

$$ {}_nC_r p^r (1 - p)^{n-r} = {}_nC_r p^r q^{n-r}, \tag{6}$$

where $q = 1 - p$ is the probability that the event will fail in a single trial.

PROOF. The probability that the event will happen in any specified combination of r trials and fail in the remaining $(n - r)$ trials is $p^r q^{n-r}$ [Art. 91, (5)], but the number of such combinations is ${}_nC_r$. Since the occurrences of exactly r times out of n trials are mutually exclusive, we find by Art. 92, that the probability in question is

$$ {}_nC_r p^n q^{n-r}.$$

By comparison with Art. 90, we note that ${}_nC_rp^rq^{n-r}$ is the $(n - r + 1)$th term of the binomial expansion (Art. 90),

$$(p + q)^n = p^n + {}_nC_1p^{n-1}\,q + {}_nC_2p^{n-2}\,q^2 + \dots \\ + {}_nC_rp^{n-r}\,q^r + \dots + q^n. \qquad (7)$$

Sometimes we are interested in the chance that an event will occur at least r times in n trials. Such is the case in undertaking to win at least two games out of three games which are to be played. This can be done by winning exactly two or all three games.

In the general case, the event happens at least r times if it happens exactly n, $n - 1$, $n - 2$, . . ., or r times in n trials. Therefore, the probability that an event of probability p will happen at least r times in n trials is

$$p^n + {}_nC_{n-1}\,p^{n-1}q + {}_nC_{n-2}\,p^{n-2}q^2 + \dots + {}_nC_rp^rq^{n-r}. \qquad (8)$$

This expression is simply the first $n - r + 1$ terms of the binomial expansion.

PROBLEMS

1. From a bag containing 5 white and 9 black balls, 4 balls are drawn. Find the probability (1) that all are white; (2) that 2 are white and 2 are black? *Ans.* (1) $\frac{5}{1001}$; (2) $\frac{360}{1001}$.

2. If 7 coins are tossed, what is the chance of three heads and four tails? *Ans.* $\frac{35}{128}$.

3. Toss 7 coins 128 times and keep a record of the number of cases of 3 heads. Then compare the relative frequency of 3 heads with the results in problem 2.

4. If seven coins are tossed, what is the probability for either three or four heads?

5. What is the probability for a sum eleven in a throw of two dice?

6. What is the probability for a sum six in a throw of two dice?

7. In a throw of seven coins, what is the probability of at least one head? *Ans.* $\frac{127}{128}$.

8. Find approximately the number of coins which must be thrown to give a probability of $^{999}/_{1000}$ that not all are heads? Hint: Use logarithms.

9. In a lottery of 100 tickets, there are 5 prizes of \$100, 10 of \$50, and 10 of \$20. Find the average value of the expectation of the holder of a ticket? *Ans.* \$12.

10. The probability that A can solve a problem is $\frac{1}{2}$, that B can solve it is $^{2}/_{3}$. What is the probability that both can solve it? What is the probability that neither can solve it?

11. A bag contains eleven balls numbered 0, 1, 2, . . . 10. A person drawing a ball is to be given the number of dollars indicated by the number on the drawn ball. What is the value of his expectation? What is the value of the expectation if the 10 is replaced by another 0?

12. From a census summary (Yule, *Theory of Statistics*, p. 159), of ages of husbands and their wives, it is found that out of 5,317,000 cases, there are 277,000 husbands of ages 60–65, and there are 226,000 wives of ages 60–65. What is the probability, correct to three significant figures that a wife taken at random is of age 60–65? That a husband taken at random would be of age 60–65? If marriage were independent of age, what would be the probability that two married persons are both of ages 60–65? Estimate the number of cases under independence out of the 5,317,000 which we should expect of a husband of age 60–65 with a wife aged 60–65? The actual number given by statistics of husbands aged 60–65 with wives of ages 60–65 is 101,000. How many more is this than is expected under independence? Explain the meaning of the large difference.

13. A machinist works 312 days in a year. If the probability of meeting with an accident on any particular day is $\frac{1}{1200}$, what is the probability that he will entirely escape accident for a year? HINT: Use logarithms. *Ans.* .7700.

14. If, in the long run, one vessel out of every 100 is damaged in a particular voyage, find the probability that of 5 vessels expected all will arrive safe. (2) That at least 4 will arrive safe.

15. In the long run, A speaks the truth 9 times out of 10, and B 4 times out of 5. They agree in the assertion that from a bag containing 10 balls of different markings, a particular one was drawn. Find the probability of the truth of the statement.

94. Probabilities of life. One of the most important fields of application of probabilities is in the chances of life and death. The problems include life insurance, life annuities, old age pensions, the valuation of life estates and of inheritance taxes based on life estates.

95. Principle of mutuality in insurance. Life insurance is possible on a sound basis when, and only when, a large group

of individuals is together in one organization distributing losses under some principle of mutuality.

Certain assessment associations have attempted to extend the principle of mutuality to the point of charging the same for a year's insurance without regard to the age of the insured. Such plans of insurance have been demonstrated to be impractical. The principle of mutuality which has been found to be practical and enduring in life insurance is that each man of the same age shall be charged the same amount for a given amount of insurance on his life. That is to say, it is practical in life insurance to assume that the individuals of a group of men of the same age are equally likely to die within a specified time.

96. Mortality table. If a large number of persons, say 100,000, could be traced from birth or from any youthful age, such as 10, until the date of death of each person, and a record made of the number living at each age x, and of the number dying between the ages of x and $x + 1$, the resultant record would be a mortality table.

It has not been found practicable in dealing with mortality to trace the individuals of a large group from birth or from an early age to death, and it is by no means necessary for the construction of a mortality table. Simpler methods have been devised for accomplishing the same end. It is, however, beyond the scope of this book to enter upon a consideration of the methods of preparing a mortality table from the statistics of deaths among persons exposed within each given year of age. For the present, we simply accept an existing mortality table in its finished form as the basis for calculations on problems of life insurance, life annuities, old age pensions, and the valuation of life estates.

There is a standard notation for dealing with the problems of mortality. Thus l_x is the number living at the age x, d_x the number dying in the age interval x to $x + 1$, p_x is the probability that a person aged x will live one year, q_x is the probability that a person aged x will die within one year.

The American Experience Table of Mortality is used for

practically all old line life insurance written at the present time in the United States. The following form giving part of this table will make clear the meaning of each number in the table. The complete table is given as Table IX at the end of the book.

AMERICAN EXPERIENCE TABLE OF MORTALITY

Age x	Number Living at Age $x = l_x$	Number Dying Aged x to $x+1 = d_x$	Probability of Living a Year from Age x to $x+1 = p_x$	Probability of Dying in Year from Age x to $x+1 = q_x$
10	100,000	749	.992510	.007490
11	99,251	746	.992484	.007516
12	98,505	743	.992457	.007543
13	97,762	740	.992421	.007569
14	97,022	737	.992404	.007596
15	96,285	735	.992366	.007634
88	2,146	744	.653308	.346692
89	1,402	555	.604137	.395863
90	847	385	.545455	.454545
91	462	246	.467534	.532466
92	216	137	.365741	.634259
93	79	58	.265823	.734177
94	21	18	.142857	.857143
95	3	3	.00000	1.00000

In this table the number living at age 10 is given as 100,000. This is simply a convenient round number called the **radix** of the table. In some mortality tables, the radix selected is 10,000 and in others it is 1,000,000. If a population has the same death rate as that given by the American Experience Table, then out of 100,000 persons at age 10, the number of deaths between 10 and 11 would be 749.

The probability q_{10} that a child age 10 will die within a year is

$$q_{10} = \frac{749}{100,000} = .00749.$$

The probability p_{10} that a child aged 10 will live a year is

$$\frac{99,251}{100,000} = .99251.$$

EXERCISE. Construct with the mortality rates of the American Experience Table (Table IX), a mortality table with 10,000 as the radix at age 20.

97. Relations among l_x, d_x, p_x, and q_x. Since l_x is the number living at age x, and d_x is the number dying between age x and $x + 1$, we have

$$d_x = l_x - l_{x+1}.$$

Hence,

$$l_{x+1} = l_x - d_x.$$

Similarly, the number of deaths within any number of consecutive years may be found from the mortality table by taking the number living at the end of the time from the number living at the beginning of the time. Thus, for n years

$$l_x - l_{x+n} = d_x + d_{x+1} + \dots + d_{x+n-1}. \tag{9}$$

When $x + n$ exceeds the oldest age in the tables,

$$l_{x+n} = 0$$

and (9) becomes

$$l_x = d_x + d_{x+1} + d_{x+2} + d_{x+3} + \dots \text{ to end of table.} \tag{10}$$

This equality (10) is obvious from the fact that each person living must die, and (10) simply states that the number living at age x is equal to the number of deaths from age x to the end of the table.

EXERCISES

1. Verify in the American Experience Table (p. 272), that $l_{10} - l_{15} = d_{10} + d_{11} + d_{12} + d_{13} + d_{14}$.

2. Verify that $l_{89} = d_{89} + d_{90} + d_{91} + \ldots$ to end of table.

3. The probability p_x that a person aged x will live one year is given by

$$p_x = \frac{l_{x+1}}{l_x}. \tag{11}$$

The probability q_x that a person aged x will die within a year is given by

$$q_x = \frac{d_x}{l_x}. \tag{12}$$

Verify in the American Experience Table that

$$p_{13} = \frac{l_{14}}{l_{13}},$$

and that

$$q_{13} = \frac{d_{13}}{l_{13}}.$$

4. Since any person of age x is certain that he will either live or die within a year, we have

$$p_x + q_x = 1. \tag{13}$$

Verify in the American Experience Table that

$$p_{12} + q_{12} = 1,$$

and that

$$p_{90} + q_{90} = 1.$$

98. Meanings of $_np_x$, $|_nq_x$, and $_n|q_x$. The symbol $_np_x$ means the probability that a person aged x will live n years. Thus, from the American Experience Table,

$$_3p_{10} = \frac{97{,}762}{100{,}000} = .97762,$$

and

$$_3p_{11} = \frac{97{,}022}{99{,}251} = .97745.$$

The abbreviation (x) is often used in life probabilities to mean

"a person aged x." In general, the probability that (x) will live n years is given by

$$ {}_np_x = \frac{l_{x+n}}{l_x}. \tag{14} $$

The probability ${}_np_x$ of living n years may be regarded as the compound event that consists of (x) living 1 year, $(x+1)$ living 1 year, $(x+2)$ living 1 year, and $(x+n-1)$ living 1 year. The probability of the compound happening is

$$ {}_np_x = p_x p_{x+1} p_{x+2} \cdots p_{x+n-1} = \frac{l_{x+n}}{l_x}. $$

The symbol $|_n q_x$ means the probability that (x) will die within n years. Thus, we have

$$ \begin{aligned} |_n q_x &= \frac{d_x + d_{x+1} + \ldots + d_{x+n-1}}{l_x}, \\ &= \frac{l_x - l_{x+n}}{l_x}, \qquad \text{[By Art. 97, (9)]} \\ &= 1 - \frac{l_{x+n}}{l_x}, \\ &= 1 - {}_np_x. \end{aligned} \tag{15} $$

Hence, $$ {}_np_x = 1 - |_n q_x. \tag{16} $$

The symbol ${}_n|q_x$ means the probability that (x) will die in the year after he reaches age $x+n$. It is, therefore, given by

$$ {}_n|q_x = \frac{d_{x+n}}{l_x}, \tag{17} $$

$$ = \frac{l_{x+n} - l_{x+n+1}}{l_x}, $$

$$ = {}_np_x - {}_{n+1}p_x. \tag{18} $$

EXERCISES

1. What is the probability that a child aged 10 will die in the year of its life between the ages 13 and 14? What is the probability that a child aged 10 will die within 3 years? *Ans.* .0074, .02238.

2. What is the probability that a man aged 40 will die within 3 years? What is the probability that he will die in the year after he reaches age 43?

99. Joint life probabilities. The probability that two lives, (x) and (y) will survive a year is denoted by p_{xy}. The probabilities of life (or of death) of two or more individuals are generally assumed to be independent of each other so that

$$p_{xy} = p_x p_y = \frac{l_{x+1} l_{y+1}}{l_x l_y}, \tag{19}$$

or as it is usually written

$$p_{xy} = \frac{l_{x+1:y+1}}{l_{xy}}, \tag{20}$$

where l_{xy} gives the number of pairs of persons living, one of age x and the other of age y.

Furthermore, $${}_n p_{xy} = {}_n p_x \; {}_n p_y. \tag{21}$$

When expressed in terms of the number living

$${}_n p_{xy} = \frac{l_{x+n \cdot y+n}}{l_{xy}}. \tag{22}$$

EXERCISES AND PROBLEMS

1. From the American Experience Table, calculate $p_{12:13}$, ${}_5p_{30:40}$. *Ans.* .98505; .90942.

2. Given two persons, A aged 45, and B aged 40, calculate the following probabilities from the American Experience Table of Mortality:

(*a*) That both will survive one year.
(*b*) That both will survive 10 years.
(*c*) That both will die during the first year.
(*d*) That A will survive the first year but B will not.
(*e*) That B will survive the first year but A will not.

MISCELLANEOUS PROBLEMS

(All calculations involving mortality are on the American Experience Table unless otherwise specified.)

1. Find, correct to three significant figures, the probability that a life just now aged 30—

(1) Will die between ages 40 and 41.

(2) Will die between ages 40 and 50.

Give these probabilities also in symbols. *Ans.* .00895; .0972.

2. Prove that ${}_3p_x = p_x \cdot p_{x+1} \cdot p_{x+2}$.

3. Find, correct to three significant figures, the probability that three lives aged 30, 40 and 50 will all survive 10 years.

4. Each of five boys is now 10 years old; what is the probability that all five of them will live to be 21 years old? That four of them will live to be 21?

5. A husband is aged 30 and a wife 25 at the date of their marriage; what is the probability that they will live to celebrate their golden wedding? What is the probability that at least one of them will be living 50 years after the date of their marriage? *Ans.* .0499; .414.

6. Find the following probabilities: That of two lives aged 18 and 22

(*a*) both will not survive 10 years;

(*b*) either one or both will survive 10 years.

7. Write the expressions, in terms of p_x and q_x, for the following probabilities: That out of 1000 persons of age (x)

(*a*) exactly 10 will die within a year;

(*b*) not more than 10 will die within a year.

8. An Italian nobleman, interested in gambling, had, by continued observation of a game with three dice, noticed that the sum 10 appeared more often than the sum 9. He expressed his surprise to Galileo and asked for an explanation. Find the probability of, (*a*) the sum 10, (*b*) the sum 9, and explain the nobleman's difficulty.

9. Five persons A, B, C, D, and E are of the same age, what is the probability that they will die in the order, A, B, C, D, and E? *Ans.* $\frac{1}{120}$.

10. From statistical data on a certain class of houses it is found that out of 1,000,000 such houses each exposed to risk of fire for one year there are 1000 complete losses by burning. What is the probability that such a house will be completely destroyed by fire within a year? What is the value of the expectation of a man who has such a house insured for $10,000 for one year, but whose insurance is against complete loss only?

11. From statistical data on a certain class of houses it is found that out of 1,000,000 such houses exposed to fire risk for one year, there are

1000 total losses and 3000 partial losses. We shall assume that a partial loss may be taken to involve one-third as much as a total loss. What is the probability of a partial loss by fire within a year? What is the probability of a total loss? What is the value of the expectation of a man who has such a house insured for $10,000? *Ans.* .003; .001; $20.

Discuss this expectation as the basis for the amount of an insurance premium the man should pay.

CHAPTER VIII

LIFE ANNUITIES

100. Factors involved. We have seen in Chapter I that the present value of a sum of money payable n years in the future depends upon the rate of interest which can be earned.

It frequently happens that a sum of money is payable at a future time, contingent on some person being alive at such future time. In this case, the present value depends upon the rate of interest, and also upon the probability that the person will be living. To illustrate, if two persons both in apparent good health aged 20 and 90, respectively, are each to receive \$100 upon attaining the ages of 25 and 95, respectively, the present value of the promised payment to the younger person would be relatively much greater than to the older person.

In algebraic language, we may say the present value of a sum payable in n years, contingent on a life now aged x attaining the age $x + n$, depends on the rate of interest i and on the probability ${}_np_x$ (Art. 98) that a person aged x will live n years.

As illustrated by this statement, the rate of interest and the probability of living are the fundamental factors in life annuities.

101. Pure endowment. If one is to be paid to a person now of age x upon attaining age $x + n$, we say the person has an n-year **pure endowment** of one.

From Art. 84, the value of the expectation of receiving a sum of money is the product of the sum and the probability of receiving it. Hence, the value of an n-year pure endowment of 1 is equal to the present value of 1 to be received at the end of n years, multiplied by the probability ${}_np_x$ that a person now aged x will survive n years.

Thus, if ${}_nE_x$ denotes the present value of an n-year pure endowment to a person of age x, we have

$$_nE_x = v^n {}_np_x. \tag{1}$$

Formula (1) may be derived from another viewpoint in which it is conceived that each of l_x persons living at age x as shown in a mortality table (Table IX, p. 272) is to purchase an n-year pure endowment of 1.

Let l_{x+n} represent the number of these persons who survive the period of n years. The total payments to them would be a sum l_{x+n}. The present value of the sum l_{x+n} is

$$v^n l_{x+n}.$$

The present value per person in the group of l_x is then

$$_nE_x = \frac{v^n l_{x+n}}{l_x} = v^n {}_np_x \text{ as given in (1).} \tag{2}$$

PROBLEMS

1. A father's will provides that a son now aged 20 is to receive $10,000 upon attaining the age 25. Find the present value of the inheritance, assuming interest at 3½ per cent and the American Experience Table of Mortality (Table IX).

Solution: The present value is given by

$$10{,}000 {}_5E_{20} = 10{,}000\, v^5 {}_5p_{20}.$$

The probability that the son will live to receive the money is

$$_5p_{20} = \frac{l_{25}}{l_{20}} = \frac{89032}{92637} = 0.9610846.$$

From Table II, $v^5 = 0.8419732.$

The present value to the son aged 20 is then

$$\$10{,}000 \times 0.8419732 \times 0.9610846 = \$8{,}092.07.$$

2. A boy aged 10 is to receive $12,000 upon attaining age 21. Find the present value of the inheritance on the basis of 6 per cent interest and the American Experience Table of Mortality. Solve also using Glover's * U. S. Life Tables, White Males, 1910.

* Students can obtain these valuable tables upon request from the Bureau of the Census.

102. Life annuity. A **life annuity** is a series of periodical payments during the continuance of one or more lives. The simplest form of life annuity to (x)* consists in the payment of 1 at the end of each year so long as a person now aged x lives. Such an annuity thus consists of the sum of pure endowments of 1 each year.

Using a_x to denote the life annuity just described, we have

$$
\begin{aligned}
a_x &= {}_1E_x + {}_2E_x + \dots + {}_nE_x + \dots \text{ to end of table,} \\
&= vp_x + v^2{}_2p_x + \dots + v^n{}_np_x + \dots, \\
&= v\,\frac{l_{x+1}}{l_x} + v^2\,\frac{l_{x+2}}{l_x} + \dots + v^n\,\frac{l_{x+n}}{l_x} + \dots, \\
&= \frac{vl_{x+1} + v^2l_{x+2} + \dots + v^nl_{x+n} + \dots}{l_x}. \qquad (3)
\end{aligned}
$$

The expression, life annuity, is used in this book to mean the kind of life annuity just described, unless it is expressly stated that some other form is intended.

Formula (3) may also be derived by conceiving that each of l_x persons living at age x as shown in a mortality table (Table IX) is to purchase an annuity of 1 per annum. Then at the end of one year there would be l_{x+1} survivors and the company granting the annuity would be called upon to pay a sum l_{x+1}, the present value of which is vl_{x+1}. At the end of two years, the company would have to pay out a sum l_{x+2}, the present value of which is v^2l_{x+2}. Stated in general terms, at the end of n years the company would have to pay out the sum l_{x+n}, the present value of which is v^nl_{x+n}.

The sum of these present values is

$$vl_{x+1} + v^2l_{x+2} + \dots + v^nl_{x+n} + \dots \text{ to end of table.}$$

The present value to each of l_x persons who purchases an annuity is

* For meaning of (x), see Art. 98.

$$a_x = \frac{vl_{x+1} + v^2 l_{x+2} + \ldots + v^n l_{x+n} + \ldots \text{to end of table}}{l_x}, \tag{4}$$

which is identical with formula (3).

EXERCISES AND PROBLEMS

1. Find the value of a life annuity of 1 per annum to a person now aged 90, assuming the American Experience Table of Mortality and $3\frac{1}{2}$ per cent interest.

Solution: By (3),

$$a_{90} = \frac{(1.035)^{-1} l_{91} + (1.035)^{-2} l_{92} + (1.035)^{-3} l_{93} + (1.035)^{-4} l_{94} + (1.035)^{-5} l_{95}}{l_{90}}$$

The mortality table (Table IX) gives

$$l_{91} = 462,\ l_{92} = 216,\ l_{93} = 79,\ l_{94} = 21,\ l_{95} = 3.$$

Making these substitutions, we have $a_{90} = 0.8738$.

2. A pension of $1000 per annum payable at the end of the year, is granted to a person now aged 91. What would you give as the present value of this pension on the basis of the American Experience Table of Mortality and 5 per cent interest? Solve also using Glover's U. S. Life Tables, White Females, 1910.

103. Fundamental relations. It is evident that if the ages chosen in the preceding problems had been, say twenty and twenty-one instead of ninety and ninety-one, the labor involved in their solution would have been several times as great. By a simple algebraic transformation of formula (4) it can be shown that the value of a life annuity at age x may readily be obtained from the value of a life annuity at age $x + 1$. From (4) Art. 102,

$$a_x = \frac{vl_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \ldots \text{ to end of table.}}{l_x}$$

Multiplying both numerator and denominator by l_{x+1} and separating the factor v from each term in the numerator, we have,

$$a_x = \frac{vl_{x+1}}{l_x} \cdot \frac{l_{x+1} + vl_{x+2} + v^2 l_{x+3} + \ldots \text{to end of table}}{l_{x+1}},$$

$$= vp_x \left(1 + \frac{vl_{x+2} + v^2l_{x+3} + \ldots \text{ to end of table}}{l_{x+1}}\right),$$

$$= vp_x\,(1 + a_{x+1}). \tag{5}$$

This relation is one of importance because it enables one to construct a complete table of life annuities with but little more labor than that involved in the computation of the value of a life annuity for the youngest age of the table.

EXERCISES

1. According to the American Experience Table of Mortality and 3½ per cent interest the value of a life annuity of 1 at age 40 is 16.446. Find the value of a_{39}.

SOLUTION: According to formula (5),

$$a_{39} = vp_{39}\,(1 + a_{40}),$$

$$= v.\frac{l_{40}}{l_{39}}.\ (1 + a_{40}),$$

$$= .966184 \times \frac{78106}{78862} \times 17.446 = 16.695.$$

2. Given that $a_{30} = 18.605$, find a_{29}, a_{28}, a_{27}, a_{26} and a_{25} according to the American Experience Table of Mortality and 3½ per cent interest.
Ans. $a_{25} = 19.442$, $a_{26} = 19.286$, $a_{27} = 19.124$, $a_{28} = 18.957$, $a_{29} = 18.784$.

104. Annuity due. If the first payment under an annuity is made immediately the annuity is called an **annuity due** and its present value for 1 per annum to a person of age x is represented by $\mathbf{a}_x$. Since an annuity due differs from an ordinary annuity only by an additional payment which is immediate, we have the relation,

$$\mathbf{a}_x = 1 + a_x. \tag{6}$$

105. Deferred annuity. It sometimes happens that the first payment under a life annuity is to be made after the lapse of a specified number of years, contingent upon the annuitant (x) being alive. Such an annuity is called a **deferred annuity.**

Since the first payment under an annuity is made at the end of one year, an annuity providing for first payment at the end

of n years is said to be deferred $n - 1$ years. In other words, the annuity is said to be entered upon at the end of $n - 1$ years, but the first payment is not made until one year later. Likewise if an annuity is deferred n years, the first payment is made at the end of $n + 1$ years.

If each of l_x persons purchases a life annuity of 1 per annum with provision for first payment at the end of $n + 1$ years, then l_{x+n+1} persons will receive the first payment, l_{x+n+2} the second, l_{x+n+3} the third, and so on. The aggregate present values of these payments will therefore be,

$$v^{n+1}\, l_{x+n+1} + v^{n+2}\, l_{x+n+2} + v^{n+3}\, l_{x+n+3} + \ldots \text{ to end of table.}$$

Dividing the last expression by l_x, the number of entrants, we arrive at the value of a life annuity of 1 on a person aged x deferred n years, represented by,

$$_{n|}a_x = \frac{v^{n+1} l_{x+n+1} + v^{n+2} l_{x+n+2} + \ldots \text{to the end of table}}{l_x}. \quad (7)$$

Multiplying both numerator and denominator of the right-hand side of (7) by l_{x+n} and taking out the factor v^n, we have, (see Art. 98),

$$\begin{aligned} _{n|}a_x &= \frac{v^n l_{x+n}}{l_x} \cdot \frac{v l_{x+n+1} + v^2 l_{x+n+2} + \ldots \text{to end of table}}{l_{x+n}}, \quad (8) \\ &= v^n {}_np_x \cdot a_{x+n}. \end{aligned}$$

EXERCISES AND PROBLEMS

1. Find $_{10|}a_{80}$.

2. Using the $1 + a_x$ column in Table X, find $_{5|}a_{36}$.

3. Assuming the American Experience Table of Mortality and $3\frac{1}{2}$ per cent interest, find $_{10|}a_{35}$, having given $a_{45} = \$15.087$.

4. By considering the value at age x of the expectation of an annuity of value a_{x+n} at age $x + n$, and referring to formula (1), obtain (8) without formal algebraic demonstration.

106. Temporary annuity. When the payments under a life annuity are to cease after a specified number of years, even

though the annuitant be still living, the promised payments constitute a **temporary annuity.** A temporary life annuity which ceases after n years is represented by $a_{x\overline{n}|}$.

If the first of the n payments is immediate and the last at the end of $n - 1$ years it is called a **temporary life annuity due,** sometimes termed an **immediate temporary annuity.***

To investigate the value of the temporary annuity, we shall proceed again from first principles. Assume that an annuity company issues to each of l_x persons a temporary annuity contract for 1 per annum to run n years. The payments at the end of the respective years would be l_{x+1}, l_{x+2}, l_{x+3}, . . . l_{x+n} units. The total present value of these payments would be

$$vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \ldots + v^nl_{x+n}.$$

Dividing by l_x, the number of entrants, we have,

$$a_{x\overline{n}|} = \frac{vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \ldots + v^nl_{x+n}}{l_x}. \tag{9}$$

EXERCISES AND PROBLEMS

1. Find the present value of a temporary life annuity of \$100 for three years to a man aged 50, using the American Experience Table and $3\frac{1}{2}$ per cent interest as a standard of mortality and interest.

SOLUTION: In this case, we have for the annuity of 1 per annum

$$a_{50\overline{3}|} = \frac{vl_{51} + v^2l_{52} + v^3l_{53}}{l_{50}}.$$

From Table II, $v = .966184$,

$v^2 = .933511$,

$v^3 = .901943$,

and from Table IX, $l_{50} = 69804$,

$l_{51} = 68842$,

$l_{52} = 67841$,

$l_{53} = 66797$.

* See Dawson's *Various Derived Tables* for this usage. The expression immediate annuity is used by some authors to mean an ordinary annuity as distinguished from a deferred annuity.

Substituting these values we have,

$$a_{50\,\overline{3}|} = \frac{(.966184)\,(68842) + (.933511)\,(67841) + (.901943)\,(66797)}{69804} = 2.7232.$$

Hence, for the temporary annuity of \$100 per annum for three years, we have,

$$100\,a_{50\,\overline{3}|} = 100 \times \$2.7232 = \$272.32.$$

2. An insurance company contracts to accept from a man aged 25 the payment of \$62.48 per annum in advance for 5 years if living as payment for insurance. Determine the equivalent single premium using the American Experience Table of Mortality and $3\frac{1}{2}$ per cent interest. *Ans.* \$287.43.

3. If $\mathrm{a}_{x\overline{n}|}$ represents a temporary annuity due, show that

$$\mathrm{a}_{x\overline{n}|} = 1 + \frac{vl_{x+1} + v^2 l_{x+2} + \ldots + v^{n-1} l_{x+n-1}}{l_x}.$$

4. Show that $\mathrm{a}_{x\overline{n}|} = 1 + a_{x\,\overline{n-1}|}$.

It is interesting to note the relation between a life annuity, a deferred annuity and a temporary annuity. A life annuity is obviously made up of those payments which are to be made during the first n years, and which therefore constitute a temporary annuity, and those payments which are to be made after n years and which therefore constitute a deferred annuity. We may therefore write,

$$a_x = a_{x\overline{n}|} + {}_{n|}a_x, \tag{10}$$

$$\text{or, } a_{x\overline{n}|} = a_x - {}_{n|}a_x, \tag{11}$$

$$\text{or, } {}_{n|}a_x = a_x - a_{x\overline{n}|}. \tag{12}$$

5. Find ${}_{15|}a_{35}$, having given $a_{35} = 17.6138$ and $a_{35\,\overline{15}|} = 10.7217$.

107. Commutation columns. The formulas thus far obtained for finding the values of life annuities involve a great deal of arithmetical computation. By the introduction of certain tables (see pp. 273, 274) known as **commutation columns,** we are able to obtain expressions for life annuities which are much simpler in application.

Multiplying both numerator and denominator of the right-hand side of the expression,

$$a_x = \frac{vl_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \text{to end of table}}{l_x},$$

by v^x, we obtain,

$$a_x = \frac{v^{x+1}l_{x+1} + v^{x+2}l_{x+2} + v^{x+3}l_{x+3} + \ldots \text{to end of table}}{v^x l_x}.$$

If now we define $D_x = v^x l_x$, the value of the life annuity may be written,

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \ldots \text{to end of table}}{D_x}.$$

Again, if we define $N_x = D_x + D_{x+1} + D_{x+2} + \ldots$ to end of table, we may write

$$a_x = \frac{N_{x+1}}{D_x}. \tag{13}$$

The value of the functions D_x and N_x according to the American Experience Table of Mortality and 3½ per cent interest are given in Table X, pp. 273, 274. The student should verify a sufficient number of these values to become thoroughly familar with them.

It should be mentioned that the definition of

$$N_x = D_x + D_{x+1} + D_{x+2} + \ldots \quad \text{to end of table}$$

is that generally adopted in America, although according to the notation adopted by the International Congress of Actuaries

$$N_x = D_{x+1} + D_{x+2} + D_{x+3} + \ldots \quad \text{to end of table.}$$

It is customary in America to use the open bar N to distinguish the American notation, but this is not universally done.

According to the notation of the International Congress of Actuaries,

$$a_x = \frac{N_x}{D_x}.$$

In making use of any set of functions N_x it is necessary to make sure which notation has been used in forming the values of N_x.

108. Other annuities expressed in commutation symbols. We have seen in Art. 107 that a_x can be expressed in commuta-

tion symbols. All the formulas thus far derived may similarly be expressed in terms of commutation symbols. Thus from the definition of D_x, Art. 107, the formula for a pure endowment may be written,

$$ {}_nE_x = \frac{v^n l_{x+n}}{l_x} = \frac{v^{x+n} l_{x+n}}{v^x l_x} = \frac{D_{x+n}}{D_x}. \tag{14} $$

The value of an annuity due may likewise be written

$$ \begin{aligned} \mathbf{a}_x &= 1 + {}_1E_x + {}_2E_x + {}_3E_x + \ \ldots \text{ to age limit,} \\ &= \frac{D_x}{D_x} + \frac{D_{x+1}}{D_x} + \frac{D_{x+2}}{D_x} + \frac{D_{x+3}}{D_x} + \ldots \text{ to age limit,} \\ &= \frac{D_x + D_{x+1} + D_{x+2} + D_{x+3}}{D_x} + \ldots \text{ to age limit,} \\ &= \frac{N_x}{D_x} \end{aligned} \tag{15} $$

from the definition of N_x, Art. 107.

Following the same line of reasoning, we have for the temporary life annuity,

$$ \begin{aligned} a_{x\overline{n}|} &= \frac{D_{x+1} + D_{x+2} + \ldots + D_{x+n}}{D_x}, \\ &= \frac{N_{x+1} - N_{x+n+1}}{D_x}. \end{aligned} \tag{16} $$

EXERCISES

1. Show that the temporary life annuity due $\mathbf{a}_{x\overline{n}|} = \dfrac{N_x - N_{x+n}}{D_x}$. (See exercise 4, Art. 106.)

2. Show that the deferred life annuity ${}_{n|}a_x = \dfrac{N_{x+n+1}}{D_x}$.

3. Find the values of a_{25}, a_{35}, a_{40}, and a_{60}.

4. Find the values of $a_{25\,\overline{10}|}$, $a_{25\,\overline{12}|}$, and $a_{25\,\overline{20}|}$.

5. Find the values of $\mathbf{a}_{25\,\overline{10}|}$, $\mathbf{a}_{25\,\overline{12}|}$, and $\mathbf{a}_{25\,\overline{20}|}$.

6. Find the values of ${}_{5|}a_{27}$, ${}_{10|}a_{27}$, and ${}_{20|}a_{27}$.

109. Annuities with payments m times a year. It is often provided in annuity contracts and also in life insurance contracts that the periodical payments shall be made more frequently than once a year. For example, life insurance premiums are payable weekly in the case of industrial insurance, certain insurance benefits are payable monthly throughout the life of the beneficiary, and pensions are in many cases payable monthly. It is important therefore to investigate the values of annuities payable m times a year. The simplest case is that of a life annuity payable m times a year.

The symbol $a_x^{(m)}$ is used to denote an annuity of annual rent 1 payable in m instalments of $\frac{1}{m}$ each, first payment at the end of $\frac{1}{m}$th of a year. The value of such an annuity is given by

$$a_x^{(m)} = \frac{1}{m}\left(v^{\frac{1}{m}}{}_{\frac{1}{m}}p_x + v^{\frac{2}{m}}{}_{\frac{2}{m}}p_x + v^{\frac{3}{m}}{}_{\frac{3}{m}}p_x + \ldots \text{to age limit}\right).$$

The evaluation of this expression would ordinarily involve a great deal of arithmetical computation. Moreover, the mortality table gives no exact information as to the probabilities of living fractional parts of a year. A satisfactory approximation for most purposes may, however, be obtained as follows:

We may write the deferred annuity due

$$_{0|}\mathrm{a}_x = (1 + a_x) - 0,$$

and

$$_{1|}\mathrm{a}_x = (1 + a_x) - 1.$$

By interpolation by proportional parts,

$$_{\frac{1}{m}|}\mathrm{a}_x = (1 + a_x) - \frac{1}{m},$$

$$= a_x + \frac{m-1}{m}.$$

Similarly, by interpolation,

$$_{\frac{2}{m}|}\mathrm{a}_x = (1 + a_x) - \frac{2}{m},$$

$$= a_x + \frac{m-2}{m},$$

and, in general,

$$_{\frac{k}{m}|}a_x = (1 + a_x) - \frac{k}{m},$$

$$= a_x + \frac{m - k}{m}.$$

If we consider that we have m annuities of 1 each per annum payable annually, the first payments of which fall due at the end of $\frac{1}{m}, \frac{2}{m}, \frac{3}{m}, \ldots \frac{m}{m}$ of a year, respectively, together they will provide 1 at the end of each $\frac{1}{m}$th year. Since the periodic payments thus provided are m times the periodic payments under the annuity in question, their combined values equal $ma_x^{(m)}$. We may therefore write

$$ma_x^{(m)} = \left[\left(a_x + \frac{m-1}{m}\right) + \left(a_x + \frac{m-2}{m}\right) + \quad \ldots + \left(a_x + \frac{m-m}{m}\right)\right].$$

The right-hand side of the above equation is the sum of an arithmetical progression with a common difference of $\frac{1}{m}$. (See Art. 149.) Summing the series, we get

$$ma_x^{(m)} = ma_x + \frac{m(m-1)}{2m},$$

and
$$a_x^{(m)} = a_x + \frac{m-1}{2m}. \qquad (17)$$

Applying (17) to annuities payable semiannually and quarterly, we have

$$a_x^{(2)} = a_x + \tfrac{1}{2},$$

and
$$a_x^{(4)} = a_x + \tfrac{3}{8}.$$

If the first payment is made at once, we have for the value of the annuity due with m instalments a year,

$$\mathbf{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)},$$

$$= a_x + \frac{m+1}{2m},$$

$$= \mathrm{a}_x - 1 + \frac{m+1}{2m},$$

whence
$$\mathrm{a}_x^{(m)} = \mathrm{a}_x - \frac{m-1}{2m}. \tag{18}$$

110. Deferred and temporary annuities payable m times a year. The value of a life annuity ${}_n|a_x^{(m)}$ deferred n years is that of an annuity $a_{x+n}^{(m)}$ to a person n years older, times the probability that (x) will live n years, times the discount factor v^n. That is,

$${}_n|a_x^{(m)} = v^n {}_np_x a_{x+n}^{(m)},$$

$$= \frac{v^n l_{x+n}}{l_x} a_{x+n}^{(m)},$$

hence,
$${}_n|a_x^{(m)} = \frac{D_{x+n}}{D_x} a_{x+n}^{(m)}. \tag{19}$$

Following the reasoning of Art. 106, the temporary life annuity for n years payable in m instalments a year is given by

$$a_{x\overline{n}|}^{(m)} = a_x^{(m)} - {}_n|a_x^{(m)}.$$

Substitute from (17) and (19), and we have

$$a_{x\overline{n}|}^{(m)} = a_x + \frac{m-1}{2m} - \frac{D_{x+n}}{D_x} a_{x+n}^{(m)}.$$

Substitute for $a_{x+n}^{(m)}$ by use of (17) and we have

$$a_{x\overline{n}|}^{(m)} = a_x + \frac{m-1}{2m} - \frac{D_{x+n}}{D_x}\left(a_{x+n} + \frac{m-1}{2m}\right),$$

$$= a_x - \frac{D_{x+n}}{D_x} a_{x+n} + \frac{m-1}{2m}\left(1 - \frac{D_{x+n}}{D_x}\right),$$

therefore,

$$a_{x\overline{n}|}^{(m)} = a_{x\overline{n}|} + \frac{m-1}{2m}(1 - {}_nE_x). \tag{20}$$

In the case of a deferred life annuity due, we have, by reasoning similar to that used to obtain (19),

$$_{n|}\mathrm{a}_x^{(m)} = \frac{D_{x+n}}{D_x}\,\mathrm{a}_{x+n}^{(m)}. \tag{21}$$

Similarly, for the temporary life annuity due,

$$\begin{aligned}
\mathrm{a}_{x\overline{n}|}^{(m)} &= \mathrm{a}_x^{(m)} - {}_{n|}\mathrm{a}_x^{(m)},\\
&= \mathrm{a}_x - \frac{m-1}{2m} - \frac{D_{x+n}}{D_x}\left(\mathrm{a}_{x+n} - \frac{m-1}{2m}\right),\\
&= \mathrm{a}_x - \frac{D_{x+n}}{D_x}\,\mathrm{a}_{x+n} - \frac{m-1}{2m}\left(1 - \frac{D_{x+n}}{D_x}\right),\\
&= \mathrm{a}_x - {}_{n|}\mathrm{a}_x - \frac{m-1}{2m}\left(1 - \frac{D_{x+n}}{D_x}\right),
\end{aligned}$$

whence,

$$\mathrm{a}_{x\overline{n}|}^{(m)} = \mathrm{a}_{x\overline{n}|} - \frac{m-1}{2m}\,(1 - {}_nE_x). \tag{22}$$

EXERCISES AND PROBLEMS *

1. The value of a temporary life annuity of \$120 per annum payable annually for 15 years to a person aged 35 is \$1286.60. What would be the value of an annuity of the same annual rent if paid in monthly instalments of \$10 each, first payment one month hence?

SOLUTION: According to the statement of the problem

$$120a_{35\overline{15}|} = \$1286.60.$$

Hence, $\quad a_{35\overline{15}|} = \$10.7217.$

From (20), $\quad a_{35\overline{15}|}^{(12)} = a_{35\overline{15}|} + \frac{11}{24}\,(1 - {}_{15}E_{35}).$

But, $\quad {}_{15}E_{35} = \dfrac{D_{50}}{D_{35}} = \dfrac{12498.6}{24544.7} = .5092.$

Hence, $\quad a_{35\overline{15}|}^{(12)} = 10.7217 + \frac{11}{24}\,(.4908).$

$$= 10.7217 + .2249 = 10.9466.$$

* In all problems which follow, assume the American Experience Table of Mortality and 3½ per cent interest, unless otherwise stated.

Therefore, the value of the annuity if paid in monthly instalments would be

$$120a^{(12)}_{35\,\overline{15|}} = (120)\ (\$10.9466) = \$1313.59.$$

2. The value of a life annuity of \$300 a year payable annually for 25 years to a person aged 25 is \$4485.45. What is the value of an annuity of the same annual rent on a monthly basis?

3. What is the value of the annuity in problem 2 if paid on a quarterly basis?

4. What is the value of the annuity in problem 1 if paid on a semi-annual basis?

5. Find the value of $a^{(12)}_{40\,\overline{20|}}$.

6. A life annuity contract provides for the payment of \$600 per annum for 25 years, first payment upon attaining age 60. If upon attaining age 60, the annuitant desires payments monthly, what would be the equitable amount of the monthly payments? *Ans.* \$52.10.

7. A municipal employee of Chicago is now of age 30. He is to receive a pension of \$50 per month beginning at age 55, the first pension payment to be received just 25 years from the present date. What is the present value of this pension?

111. Forborne temporary annuity due. Suppose (x) is entitled to a life annuity due of 1 per year, but forbears to draw it, with the agreement that the unpaid instalments are to accumulate as pure endowments until he is aged $x + n$. Then the annuity is called a **forborne temporary annuity due,** or **forborne immediate annuity.**

More briefly, a forborne temporary annuity due is the pure endowment which 1 per annum in advance will buy.

The value at age x of a temporary annuity due of 1 per year for n payments is

$$1 + a_{x\overline{n-1|}} = \frac{N_x - N_{x+n}}{D_x}.$$

To find the amount at age $x + n$ of the forborne temporary annuity due of 1, the value

$$\frac{N_x - N_{x+n}}{D_x}$$

at age x must be used to purchase an n-year pure endowment.

By (14) Art. 108, it follows readily that 1 at age x will purchase an n-year pure endowment of

$$\frac{D_x}{D_{x+n}}.$$

Hence, the present value $\dfrac{N_x - N_{x+n}}{D_x}$ would buy as a pure endowment an amount at age $x + n$ equal to

$$\frac{D_x}{D_{x+n}} \cdot \frac{N_x - N_{x+n}}{D_x} = \frac{N_x - N_{x+n}}{D_{x+n}}. \tag{23}$$

This amount is, then, by definition, the value of the forborne temporary annuity due taken at age x to the person still living at age $x + n$.

The function $\dfrac{N_x - N_{x+n}}{D_{x+n}}$ is one of the most useful functions in the practical work of the actuary.

EXERCISES AND PROBLEMS

1. Find the amount at age 70 of a forborne temporary annuity due of 1 per annum that is to be accumulated for a person now aged 30. *Ans.* \$164.47.

2. If \$100 per year in advance payable by a man aged 30 is to accumulate as a pure endowment for 20 years, what sum will be produced? *Ans.* \$3321.49.

MISCELLANEOUS EXERCISES AND PROBLEMS

1. By means of the commutation columns, find the value of ${}_{10}E_{35}$, ${}_{15}E_{35}$ and ${}_{20}E_{35}$.

2. Using the D_x commutation column only, find the value of a_{85} and verify by using both the N_x and D_x functions.

3. The annual premium on a whole life non-participating insurance policy at age 30 is \$24.60. What would be the equivalent single premium? *Ans.* \$482.29.

4. Find the value of ${}_{35|}a_{30}$.

5. A will provides that a person now aged 35 is to receive an income for life of $1000 per annum, first payment upon attaining age sixty-five. What is the value of the promised benefit?

6. Find the value of $a_{25\,\overline{20|}}$.

7. A life insurance policy for $10,000 provides that the proceeds shall be paid in annual instalments for twenty years certain, first payment upon due proof of the death of the insured. What is the equitable amount of the annual instalments? *Ans.* $679.82.

8. What should be the amount of the annual instalments in the above problem if the proceeds were payable throughout the life of the beneficiary, assuming that the beneficiary were twenty-five years of age at the death of the insured? *Ans.* $489.20.

9. What would be the amount of the annual instalments in problem 8 if payable for 20 years, but each payment contingent upon the beneficiary being alive? *Ans.* $729.59.

10. What would be the amount of the annual instalments in problem 8 if payable for twenty years certain and so long thereafter as the beneficiary survives? *Ans.* $466.30.

11. Find the values of $a_{40}^{(4)}$, $a_{40}^{(12)}$, $\text{a}_{40}^{(4)}$ and $\text{a}_{40}^{(12)}$.

12. A will provides that a wife shall receive an income of $100 per month so long as she lives after her husband's death, first payment immediately upon the death of the husband. If she is fifty years of age at the death of the husband, what will be the value of the promised payments at that time?

13. Find the value of $a_{25\,\overline{10|}}^{(12)}$ and $\text{a}_{25\,\overline{10|}}^{(12)}$.

14. Under the first pension plans of the Carnegie Foundation, a professor could retire on a yearly pension of half "final salary" plus $400 after 25 years of service beginning with the date of becoming assistant professor. If such a plan were carried out, what would be the present value at age 28 to a man who became an assistant professor at this age, whose "final salary" will be $5000 if he should accept the pension at the end of the assigned 25 years of service. In solving this problem, treat the pensions as if payable at the ends of the years.

15. Under the present regulations of the Carnegie Foundation, a man aged 30 will receive a yearly pension of half "final salary" plus $400 at age 70. Former regulations would have given this pension at age 65. What is the difference between the present values of the expectations if the man is to have a final salary of $5000?

16. A person aged 25 has received as an inheritance a life income of $500 per month, first payment at once. An inheritance tax of 5 per cent

on the present value is to be be paid at once. Find the amount of the tax payable.

17. A man is to receive a life annuity of $1000 per year, the first annual payment to be made on his 61st birthday. Wishing to receive a larger annuity in his old age he arranges to postpone the beginning of the annuity for 10 years. On the basis of the American Experience Table and $3\frac{1}{2}$ per cent what yearly sum will he receive if the first payment will be made on his 71st birthday?

18. A suit for damages on account of the accidental death of a railroad man 38 years old and earning $1500 per year was settled on the basis of two-thirds of the present value of the expected wages of $1500 per year during his after lifetime, computations to be made according to the American Experience Table at $3\frac{1}{2}$ per cent. What were the damages?

19. According to the rules of the Episcopal Church pension fund, a clergyman retiring at age 68 is to receive an annuity equal to one and a quarter per cent of his average stipend multiplied by the number of years he has received a stipend. If a man were ordained at age 27 and is now 45 years old, what is the present value of his pension based on the American Experience Table and $3\frac{1}{2}$ per cent, assuming that his average stipend will be $1800 and that he receives the pension in annual instalments, first instalment being paid on his 69th birthday?

20. If there had been no pension system, what amount should the clergyman in problem 19 have set aside yearly during his active service for investment and accumulation at 5 per cent in order to accumulate by age 68 an amount equal to the value of the pension at that time?

CHAPTER IX

NET PREMIUMS FOR SOME SIMPLE FORMS OF LIFE INSURANCE

112. Introduction. As explained in Art. 95, life insurance is possible on a sound basis when, and only when, a large group of persons is together in an organization or company distributing losses on the group by some principle of mutuality.

All of the insured make payments to the company, and payments are made by the company in behalf of such of the insured as suffer losses for which indemnity is to be provided by the insurance. The payments by the insured to the company are called **premiums,** or sometimes **gross** or **office** premiums. The recipient of payments by the company on behalf of one who dies is called a **beneficiary.**

The term **benefit** is often used in a popular sense to refer to the satisfaction that arises from insurance protection, but we shall ordinarily use it in a more precise sense to mean the expectation (Art. 84) of the payments by the company.

The written contract between the insured and the company is called a **policy** and the insured is often called a **policyholder.**

For some purposes in connection with policy descriptions it is convenient to measure time from the date from which a premium is charged. This is called the **policy date** or sometimes the **valuation date of issue.** The period of one year measured from the policy date is called the **first policy year,** the next is called the **second policy year** and so on.

In this book, we shall deal only with a few of the simpler forms of policies offered by legal reserve (Art. 121) life insurance companies. We shall first be concerned with the determination of premiums known as net premiums.

A **net premium** is simply an equivalent of the present value of the benefit calculated according to an assigned table of mortality and rate of interest. In other words the net premium is the

amount which the company would have to receive if the mortality exactly equaled the assumed mortality, if the interest earnings exactly equaled the assumed interest earnings, if there were no expenses involved in the conduct of the business and if death losses were paid at the end of the policy year in which death occurs.

113. Net single premium. When the present value of an insurance is expressed as a single sum, it is known as a **net single premium.** The net single premium is clearly equal to the present value of the benefit under the assumed rates of mortality and interest.

This premium for what is called a **whole life policy** * may be readily obtained by considering that a company issues a policy of 1 to each of l_x persons living at age x as shown in a mortality table, and that deaths take place according to the table.

If this be done, the company would at the end of the first policy year pay out 1 for each of the d_x deaths between ages x and $x + 1$; at the end of the second year it would pay out d_{x+1}; at the end of the third, d_{x+2} and so on. Hence, the present value of the payments by the company is

$$vd_x + v^2d_{x+1} + v^3d_{x+2} + \ldots \text{ to end of table.} \tag{1}$$

If this sum be divided by l_x, the number insured, we arrive at the present value of the insurance to each of the l_x persons, which is the net single premium at age x, denoted by A_x. Thus, we have

$$A_x = \frac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \ldots \text{ to end of table}}{l_x}. \tag{2}$$

The American Experience Table of Mortality has become the standard for the calculation of net premiums and for the valuation of policies in the United States. Unless otherwise stated we shall in all our exercises and problems on life insurance assume this table given on p. 272. We shall also assume interest at 3½ per cent unless we make a definite statement as to a different rate of interest.

* A whole life policy is one in which the sum insured is payable at death and at death only.

PROBLEMS

1. Find the net single premium for a whole life policy to insure a life aged 92 for \$1000.

FORM FOR SOLUTION: In this case, we have by formula (2),

$$1000A_x = 1000\frac{(1.035)^{-1}\,d_{92} + (1.035)^{-2}\,d_{93} + (1.035)^{-3}\,d_{94} + (1.035)^{-4}\,d_{95}}{l_{92}}.$$

From the mortality table, p. 272.

$$d_{92} = 137,\ d_{93} = 58,\ d_{94} = 18,\ d_{95} = 3.$$

Substitution of these values gives

$$1000\,A_x =$$

2. Outline the solution of the problem of finding the net single premium for a whole life insurance for \$1000 for a life aged 40.

114. Commutation columns C and M. Multiply the numerator and denominator of the right-hand member of (2) Art. 113, by v^x. This gives

$$A_x = \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + v^{x+3}d_{x+2} + \ldots \text{to end of table}}{v^x l_x}. \quad (3)$$

Now introduce the commutation symbol

$$C_x = v^{x+1}d_x,$$

and we have

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \ldots \text{to end of table}}{D_x} \quad (4)$$

$$= \frac{M_x}{D_x}, \quad (5)$$

where the commutation symbol M_x is the sum of the column C_x from age x to the oldest age of the table inclusive.

The functions C_x and M_x for the American Experience Table of Mortality and 3½ per cent interest are shown in Table X.

EXERCISES AND PROBLEMS

1. Find the net single premium for a whole life insurance of \$5000 on a life aged 25.

FORM FOR SOLUTION: By formula (5)

$$A_{25} = \frac{M_{25}}{D_{25}}$$

$$=$$

The net single premium for \$5000 * is then

$$5000\ A_{25} =$$

2. Same as exercise 1 except that the person is of age 50. *Ans.* \$2542.45.

3. What is the increase in the net single premium for a whole life insurance of \$1000 when the age changes from, (*a*) 20 to 21; (*b*) 50 to 51?

115. Relations between net single premium A_x and present value of life annuity a_x. The close relation between insurance and annuities may be demonstrated by expressing A_x in terms of a_x. Thus, from (2) Art. 113,

$$A_x = \frac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \ldots \text{ to end of table}}{l_x},$$

$$= \frac{v\,(l_x - l_{x+1}) + v^2(l_{x+1} - l_{x+2}) + v^3(l_{x+2} - l_{x+3}) + \ldots}{l_x},$$

$$= \frac{vl_x + v^2l_{x+1} + v^3l_{x+2} + \ldots}{l_x} - \frac{vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \ldots}{l_x},$$

$$= v\left(1 + \frac{vl_{x+1} + v^2l_{x+2} + \ldots}{l_x}\right) - a_x, \text{ by (3) Art. 102, or}$$

$$A_x = v\,(1 + a_x) - a_x. \tag{6}$$

This is an important relation, but it is still possible to trans-

* It is common practice to calculate the net premium on \$1000 to the nearest cent, and then to find the premium for any amount of insurance by multiplying this premium for \$1000 by the number of thousands of insurance. We shall follow this practice in our problems.

form it into a form that may be more readily explained verbally. For this purpose we introduce the factor d (Art. 19) where

$$d = iv = \frac{i}{1+i} = 1 - v.$$

Thus, from (6), we obtain

$$\begin{aligned} A_x = \frac{1+a_x}{1+i} - a_x &= \frac{1 - ia_x}{1+i}, \\ &= \frac{1}{1+i} - \frac{ia_x}{1+i}, \\ &= v - da_x, \\ &= 1 - d - da_x, \end{aligned}$$

whence $$\mathbf{A_x = 1 - d\,(1 + a_x).} \tag{7}$$

It is a valuable exercise to think through this relation so as to be able to write it down without any formal algebraic derivation, but with the support of the following reasoning.

The sum insured is 1, and, if it were payable at once, its value would be simply 1, but since it is not payable until the end of the year of death of a person now aged x, we must deduct the value of the interest on one per annum throughout the life of (x). The value at the beginning of the year of the interest for each year on 1 is d. Hence, the present value of the interest on 1 throughout the life of a person aged x is

$$d(1 + a_x).$$

Deducting this present value from 1, we obtain (7).

EXERCISES

1. A life annuity of \$1 per annum at age of 20 years has a present value \$20.144 on a $3\frac{1}{2}$ per cent interest basis. What is the net single premium at age of 20 years for a whole life insurance of \$1 with the same mortality table and interest rate? What is the net single premium at age of 20 years for an insurance of \$1000?

2. Same as exercise 1 except age 25 years. In this case $a_{25} = \$19.442$.

3. Express a_x in terms of v and A_x; in terms of d and A_x; in terms of i and A_x.

116. Annual premiums. Life insurance premiums are most frequently paid in equal annual instalments. The annual premiums may continue throughout the life of the policyholder, or they may be limited to a definite number of years. Premiums may also be paid semiannually, quarterly, or monthly; and, in the case of industrial insurance, weekly.

When the instalment premiums on a whole life policy continue throughout the life of the insured, the policy is called an **ordinary life policy.** When the payments are limited to a certain number of years, the policy is called a **limited payment life policy,** or an **n-payment life policy,** if n is the limit on the number of annual payments.

The **net annual premium** is that sum which, if payable at the beginning of each policy year for life or for a limited number of years, is the equivalent of the net single premium.

The payments of the net annual premium for an ordinary life policy constitute therefore an annuity due payable by the policyholder to the company.

If P_x represents the net annual premium for an ordinary life policy at age x for an insurance of 1, we have,

$$P_x\,(1 + a_x) = A_x. \tag{8}$$

Hence

$$P_x = \frac{A_x}{1 + a_x}. \tag{9}$$

By (5) Art. 114, $A_x = \dfrac{M_x}{D_x}.$

By (15) Art. 108, $1 + a_x = \dfrac{N_x}{D_x}.$

Hence,

$$P_x = \frac{M_x}{N_x}. \tag{10}$$

The annual premium, P_x, for an ordinary life policy may also be expressed in terms of annuity values. We have seen in Art. 115 that

$$A_x = 1 - d\,(1 + a_x).$$

By substituting this value of A_x in (9),

$$P_x = \frac{1 - d\,(1 + a_x)}{1 + a_x};$$

that is,

$$\boldsymbol{P_x = \frac{1}{1 + a_x} - d.} \tag{11}$$

The payments of the net annual premium, ${}_nP_x$, for an n-payment life policy constitute a temporary annuity due of n payments. Using $1 + a_{x\,\overline{n-1|}}$ for a temporary life annuity due of 1 per annum for n payments, we have

$${}_nP_x\,(1 + a_{x\,\overline{n-1|}}) = A_x.$$

Hence,

$${}_nP_x = \frac{A_x}{1 + a_{x\,\overline{n-1|}}}. \tag{12}$$

Expressed in commutation symbols, we have by (16) Art. 108,

$$1 + a_{x\,\overline{n-1|}} = 1 + \frac{N_{x+1} - N_{x+n}}{D_x} = \frac{N_x - N_{x+n}}{D_x}.$$

Hence,

$$\boldsymbol{{}_nP_x = \frac{M_x}{N_x - N_{x+n}}.} \tag{13}$$

EXERCISES

1. What is the net annual premium for an ordinary life policy of \$5000 on a life aged 25? *Ans.* \$75.50.

SOLUTION: By formula (10), we have for the net annual premium,

$$P_{25} = \frac{M_{25}}{N_{25}} = \frac{11631.1}{770113} = .01510.$$

For a policy of \$5000, we have then for the net annual premium

$$5000 \cdot P_{25} = \$75.50.$$

2. What is the net annual premium for a twenty payment life policy for \$5000 issued at age 25? *Ans.* \$112.65.

3. Find the net annual premium for a ten payment life policy for \$5000 issued at age 50. *Ans.* \$316.00.

4. Give a verbal interpretation of formula (11) by noting that if 1 were payable immediately, its value per year in advance throughout the life of (x) would be $\dfrac{1}{1 + a_x}$.

5. By taking from Table X the values of the annuities due, find from formula (11), the annual premiums for ages 30 and 40 respectively for an ordinary life insurance of \$1000.

6. Give a verbal interpretation of the formula $A_x = v(1 + a_x) - a_x$.

117. Net single premium for term insurance. A **term policy** is one in which the face is payable in the event of death within a stated term, and only on condition that death occurs within that term. The stated term varies widely. Thus, we may have a one year term policy, a five year term policy, a ten year term policy, a twenty year term policy, a thirty year term policy and so on.

The premium for a term policy for n years may be obtained by considering that a company issues a policy of 1 to each of l_x persons living at age x as shown in the mortality table. The present value of the payments by the company is

$$vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots + v^nd_{x+n-1}. \tag{14}$$

The net single premium for a term insurance for n years taken at age x is denoted by either of the symbols $|_nA_x$ or $A^{1}_{x\overline{n}|}$. It seems that the latter is more commonly used and we shall adopt it for use in this book.

If the sum (14) be divided by l_x, the number insured, we have the present value of the insurance to each of the l_x persons, which is the net single premium for an n-year term insurance of 1 at age x. Thus,

$$A^{1}_{x\overline{n}|} = \frac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \cdots + v^nd_{x+n-1}}{l_x}. \tag{15}$$

Multiplying numerator and denominator of the right-hand

member of (15) by v^x, and substituting the commutation symbols C_x, and D_x, we have,

$$A^{\ 1}_{x\overline{n|}} = \frac{C_x + C_{x+1} + C_{x+2} + \ldots + C_{x+n-1}}{D_x}. \tag{16}$$

The numerator of the right-hand member of (16) is equal to the numerator of (4) Art. 114 less the sum

$$M_{x+n} = C_{x+n} + C_{x+n+1} + \ldots \text{ to end of table.}$$

But the numerator of (4) Art. 114, is equal to M_x.

Hence,
$$A^{\ 1}_{x\overline{n|}} = \frac{M_x - M_{x+n}}{D_x}. \tag{17}$$

When the term insurance is for 1 year only the net premium is sometimes called the **natural premium.** Its value is given by making $n = 1$ in (17). This gives

$$A^{\ 1}_{x\overline{1|}} = \frac{M_x - M_{x+1}}{D_x} = \frac{C_x}{D_x}. \tag{18}$$

The expression natural premium is convenient in describing the basis of valuation of policies treated in the next chapter.

EXERCISES

1. Find the net single premium for a term insurance for \$12,000 for 10 years for a man aged 22.

Form for Solution: By formula (17) the required single premium is

$$12{,}000\, A^{\ 1}_{22\,\overline{10|}} = 12{,}000\, \frac{M_{22} - M_{32}}{D_{22}}$$

$$=$$

2. What are the natural premiums at ages 30, 35, 40 and 45 for an insurance of \$1000?

118. Annual premium for term insurance. The payments of the net annual premium for term insurance constitute a temporary annuity due as in the case of the annual premium on a limited payment life policy [see (12), Art. 116].

In the usual notation the net annual premium for an n-year term policy of 1 on a life aged x is denoted by $P^1_{x\overline{n}|}$ or $_{|n}P_x$. Since such premiums constitute a temporary annuity due payable to the company, we have

$$P^1_{x\overline{n}|}(1 + a_{x\overline{n-1}|}) = A^1_{x\overline{n}|}.$$

Hence,
$$P^1_{x\overline{n}|} = \frac{A^1_{x\overline{n}|}}{1+a_{x\,\overline{n-1}|}} = \frac{A^1_{x\overline{n}|}}{\mathbf{a}_{x\overline{n}|}}. \tag{19}$$

Expressed in commutation symbols, we have by exercise 1, Art. 108 and (17) Art. 117.

$$P^1_{x\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \tag{20}$$

EXERCISES

1. Find the net annual premium for a five year term insurance of \$5000 on a life aged 25.

Form for Solution: By (20), Art. 118,

$$P^1_{25\,\overline{5}|} = \frac{M_{25} - M_{30}}{N_{25} - N_{30}}$$

$$=$$

when values for commutation symbols are substituted from Table X.

For a policy of \$5000, the net annual premium would be

$$5000\, P^1_{25\,\overline{5}|} =$$

2. Find the net annual premium for a term policy of \$10,000 issued on a life aged 30, the policy to terminate at age 65.

119. Net single premium for endowment insurance. An **endowment insurance** provides for the payment of the face of the policy in event of the death of the insured within a certain period, called the **endowment period,** and also provides for the payment of the face of the policy at the end of the endowment period, provided the insured be living. By definition, an endowment insurance of 1 for n years may

obviously be looked upon as a term insurance of 1 for n years plus an n-year pure endowment of 1.

Hence, if we denote the single premium for an endowment insurance of 1 by $A_{x\overline{n}|}$, we have by Arts. 101, and 117,

$$A_{x\overline{n}|} = A^{1}_{x\overline{n}|} + {}_nE_x. \tag{21}$$

But by (17) Art. 117, $$A^{1}_{x\overline{n}|} = \frac{M_x - M_{x+n}}{D_x},$$

and by (14) Art. 108 $${}_nE_x = \frac{D_{x+n}}{D_x}.$$

Hence,

$$A_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \tag{22}$$

120. Annual premium for endowment insurance. Here again, as in the cases of limited payment life insurance and term insurance, the payments of the net annual premium for r years for an n year endowment insurance of 1 constitute a temporary annuity due that is equivalent to the net single premium $A_{x\overline{n}|}$.

Hence, denoting the net annual premium payable for r years by ${}_rP_{x\overline{n}|}$, we have

$${}_rP_{x\overline{n}|} = \frac{A_{x\overline{n}|}}{1 + a_{x\overline{r-1}|}} = \frac{A_{x\overline{n}|}}{\mathbf{a}_{x\overline{r}|}}. \tag{23}$$

Expressed in commutation symbols, we have by exercise 1, Art. 108, and (22) Art. 119,

$${}_rP_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+r}}. \tag{24}$$

In particular, if $r = n$ we have

$$P_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}. \tag{25}$$

EXERCISES AND PROBLEMS

1. Find the net annual premium on a twenty year endowment policy for $10,000 purchased at age 25.

FORM FOR SOLUTION: By (25), Art. 120,

$$P_{25\,\overline{20|}} = \frac{M_{25} - M_{45} + D_{45}}{D_{25}},$$

$$=$$

when values of commutation symbols are substituted from Table X.

2. A man aged 45 plans to provide for his old age and for the creation of an estate if he dies before age 65, by taking $25,000 of twenty year endowment insurance. Find the net annual premium. What would the premium be if term insurance only had been provided for the same period? *Ans.* $1077.00; $434.25.

3. Find the net annual premium for a twenty payment endowment insurance maturing at age 65 on a life aged 25 at date of issue.

MISCELLANEOUS PROBLEMS

1. Find the net annual premium for an ordinary life policy for $1000 taken at age 20. *Ans.* $13.48.

2. Same as problem 1 for an endowment policy maturing at age 85. What is the difference in annual premiums between the ordinary life and the endowment policy maturing at age 85?

3. Same as problem 1 for an endowment policy maturing in 10 years. What is the difference in annual premiums between the ordinary life and the endowment policy maturing at age 30? How would you by common sense reasoning account for the greater difference in this case than in that of problem 2? *Ans.* $86.30; $72.82.

4. A whole life insurance policy for $1000 taken at age 25 provides that the insurance of the first year is term insurance for the year, and thereafter the insurance is under an ordinary life policy. What is the net premium for the first year, and for any subsequent year? *Ans.* First year, $7.79; subsequent year, $15.48.

5. A whole life policy for $1000 taken at age 25 provides that the insurance of the first year is term insurance for the year, and that of subsequent years is a 19 payment life insurance, so that the insurance is paid up in 20 payments in all. What is the net premium of the first year and of any subsequent year? *Ans.* First year, $7.79; subsequent year, $23.68.

6. A thirty year endowment policy for $1000 taken at age 25 provides that the net premium of the first year is simply that for term insurance at

age 25, and the net annual premium for the subsequent 29 years is that for a 29 year endowment policy. What is the net premium for the first and for subsequent years?

7. The net single premium for paid up life insurance at age 45 is $456.00 per $1000 of insurance. Find the net annual premium for a twenty payment life policy at age 25, and also for a twenty year pure endowment of $544.00. Find how the sum of these net premiums compares with the net annual premium for a twenty year endowment insurance for $1000 at the same age.

8. Prove $M_x = vN_x - N_{x+1}$.

9. Prove $A_x = \dfrac{P_x}{P_x + d}$.

10. What is the net single premium for an insurance in which the beneficiary receives $1000 at the death of the policyholder and $1000 per year for the subsequent nine years?

11. Find the net annual premium for the insurance of problem 10.

12. Express each of the symbols A_x, a_x, P_x in terms of the other two.

13. What is the net annual premium on a whole life policy for $1000 issued at age 35 with the condition that on the death of the insured the amount is to be retained by the company to accumulate at compound interest at 5 per cent for 10 years?

14. What is the net annual premium on a twenty year endowment policy issued at age 40 and payable in ten equal annual instalments of $1000, the first instalment due at the end of the year of death or at the maturity of the endowment if the insured be then living?

CHAPTER X

VALUATION OF LIFE INSURANCE POLICIES

121. Meaning of reserves. An examination of any mortality table will disclose the fact that, except for the very early years of life, the probability of dying within a year increases with advance in age. Accordingly, the net premium for one year term insurance, called the natural premium (Art. 117), becomes greater with advancing age. If a person of age x purchases insurance for a term of years or for the whole of life at a level net premium payable for a term of years or for the whole of life, it is evident that in the early years of insurance, the insured pays a net premium greater than the natural premium, because in the later years the level net premium will not be adequate to pay the death claims due to advancing age. Hence, that part of the level premiums of the early years not required for mortality purposes is held and improved at interest by the company to meet the heavier mortality of later years. The amount so held is a liability of the company known as the **reserve** on its policies, or as the **value** of its policies.

From this explanation of the meaning of the reserve it should be clear that the reserve is not, as assumed by some, a provision against unexpected losses or contingencies. Far from a provision against unforseen losses, the reserve is a liability of the company to its policyholders in a way somewhat similar to the liability of a bank to its depositors. However, they differ in that the reserve on a policy does not belong to the individual policyholder in the sense that he can withdraw it at will as he could a bank deposit. This point will be further considered in Art. 122.

122. Prospective method of valuation. We may look at the reserve from another viewpoint which leads to what is known as the **prospective method** of valuation of policies.

To explain this method, let us consider first an ordinary life insurance of 1 on each of a group of lives aged x. At the end of the nth policy year, such a person is of age $x + n$. The value of the benefit per person at age $x + n$ is simply the net single premium

$$A_{x+n}.$$

But the net annual premium P_x payable to the company by each of the insured is still to be paid throughout the life of each person aged $x + n$.

This life annuity of P_x per year at age $x + n$ has a value

$$P_x(1 + a_{x+n}).$$

The difference

$$_nV_x = A_{x+n} - P_x(1 + a_{x+n}), \tag{1}$$

is the reserve on an insurance of 1 at the end of the nth policy year.

This value $_nV_x$ is known as the **terminal reserve** of the nth policy year on an insurance of 1.

We may state the broad general principle of prospective valuation by saying that whatever the form of insurance, *the terminal reserve of the nth policy year is equal to the net single premium for the benefit at attained age $x + n$, less the present value as of age $x + n$ of the future net premiums.*

While the sum of the individual policy reserves correctly represents the reserve liability of the company in the aggregate, it is important to emphasize that the reserve should not be considered as belonging to the individual policyholders as bank deposits belong to individual depositors. It should always be remembered that the insurance business is one of averages, and that in a certain sense the actual reserve on an individual policy may be very different from the theoretical reserve. For example, if a policyholder has become impaired in health so that he is no longer an average risk, the single premium that should be charged for one of his class will be much greater than that for an average risk and the value of the future net premiums based upon his probability of living would be much less than for

an average risk. This condition would therefore result in giving a larger reserve liability than that for an average life. On the other hand, if a policyholder were above the average in health, the actual reserve would be less than the theoretical reserve.

This explanation is given in order that the student may see clearly the meaning of the theoretical reserve and form a conception of its value on a policy compared to the sum which a withdrawing policyholder should be paid. The sum which a company will pay the withdrawing policyholder upon the surrender and cancelment of his contract is called the **surrender value** of the policy. It is usually a large part of the reserve, but the above explanation should make it clear that the company is justified in not giving the entire amount of the theoretical reserve on a policy as a surrender value.

EXERCISES

1. Find the terminal reserve of the tenth policy year on an ordinary life policy of $5000 taken at age 25.

SOLUTION: By (1)

$${}_{10}V_{25} = A_{35} - P_{25}\,(1 + a_{35}).$$

By Table X, $A_{35} = 0.37055.$

By Table X, $1 + a_{35} = 18.6138.$

By (10), Art. 116, $P_{25} = 0.0151031.$

Substitution of these values gives

$${}_{10}V_{25} = 0.08942^*.$$

Hence, $5000\,{}_{10}V_{25} = \$447.10.$

2. Find the terminal reserve of the fifteenth year on an ordinary life policy of $8000 taken at age 35. *Ans.* $1753.20.

3. Show that the terminal reserve of the nth policy year on a single premium policy taken at age x is simply the net single premium at the attained age $x + n$ of the insured.

* In the calculation of reserves, it is common practice to find the reserve on $1000 of insurance to the nearest cent. Then the reserve on any amount is found by multiplying this reserve on $1000 by the number of thousands of insurance. We shall follow this practice in our problems.

123. Transformation and verbal interpretation. Remembering by (8) Art. 116 that

$$A_x = P_x(1 + a_x),$$

we may transform formula (1) as follows:

$$\begin{aligned} {}_nV_x &= A_{x+n} - P_x(1 + a_{x+n}), \\ &= P_{x+n}(1 + a_{x+n}) - P_x(1 + a_{x+n}), \end{aligned}$$

whence $$\boldsymbol{{}_nV_x = (P_{x+n} - P_x)(1 + a_{x+n})}. \tag{2}$$

This formula readily lends itself to verbal interpretation. If a man aged $x + n$ were to effect a whole life insurance, the net annual premium would be represented by P_{x+n}, but having effected it at age x at a net annual premium represented by P_x, he is relieved of paying $(P_{x+n} - P_x)$ annually for the remainder of his life. The present value of the annual payments from which he is relieved by having entered at age x is therefore equal to

$$(P_{x+n} - P_x)(1 + a_{x+n}),$$

which represents the policy value or reserve.

We may also express the value of the reserve in terms of annuities. From (7) and (11) of the preceding chapter,

$$A_x = 1 - d(1 + a_x),$$

and $$P_x = \frac{1}{1 + a_x} - d.$$

Substituting in (1), we have

$$\begin{aligned} {}_nV_x &= A_{x+n} - P_x(1 + a_{x+n}), \\ &= 1 - d(1 + a_{x+n}) - \left(\frac{1}{1 + a_x} - d\right)(1 + a_{x+n}), \\ &= 1 - d(1 + a_{x+n}) - \frac{1 + a_{x+n}}{1 + a_x} + d(1 + a_{x+n}), \\ &= 1 - \frac{1 + a_{x+n}}{1 + a_x}. \end{aligned} \tag{3}$$

EXERCISES

1. Having available a table of life annuities such as are given in Table X, find the terminal reserve of the tenth policy year on an ordinary life policy for \$5000 issued at age 35. *Ans.* \$678.80.

2. Having available a table of life annuities and of net single premiums for whole life insurance, as given in Table X, calculate the required annual premium, and use formula (2) to find the eighth year terminal reserve on an ordinary life policy of \$4000 issued at age 30. *Ans.* \$340.72.

124. Reserve for limited payment life insurance. In Art. 122, we stated that whatever the form of insurance, the terminal reserve of the nth policy year is equal to the net single premium for the benefit at the attained age of the insured less the present value of the future net premiums. This is a fundamental principle that should be well fixed in the mind of the student.

Hence, denoting the terminal reserve of the nth policy year on an m-payment life policy by ${}_{n:m}V_x$, we have if $n < m$,

$$ {}_{n:m}V_x = A_{x+n} - {}_mP_x(1 + a_{x+n\,\overline{m-n-1}|}), \tag{4}$$

where m is the number of years in the premium paying period. If $n \overline{\gtrless} m$, the nth year terminal reserve is simply equal to the net single premium

$$A_{x+n}.$$

Formula (4) may be expressed in another useful form by the substitution

$$A_{x+n} = {}_{\overline{m-n}}P_{x+n}(1 + a_{x+n\,\overline{m-n-1}|}).$$

This gives

$$ {}_{n:m}V_x = ({}_{\overline{m-n}}P_{x+n} - {}_mP_x)\,(1 + a_{x+n\,\overline{m-n-1}|}), \tag{5}$$

which is analogous to (2).

The equation (5) can be written at once without formal derivation from the fact that the insured who has carried the m payment life policy for n years at annual premium ${}_mP_x$ should have to pay

$${}_{\overline{m-n}}P_{x+n} - {}_mP_x$$

less per year for the remaining $(m - n)$ years than a man who at age $x + n$ takes an $(m - n)$ payment life policy. This annual difference as a temporary annuity due of $(m - n)$ payments has a value equal to the reserve.

EXERCISES

1. Find the terminal reserve on a \$1000 twenty-payment life policy taken at age 25 for the tenth policy year.

FORM FOR SOLUTION: By (4), Art. 124,

$$_{10:\,20}V_{25} = A_{35} - {}_{20}P_x\,(1 + a_{35\,\overline{9|}}).$$

By Table X, $\qquad A_{35} =$

By (13), Art. 116, $\qquad {}_{20}P_{25} = \dfrac{M_{25}}{N_{25} - N_{45}} =$

By (16), Art. 108, $\quad 1 + a_{35\,\overline{9|}} = 1 + \dfrac{N_{36} - N_{45}}{D_{35}} = \dfrac{N_{35} - N_{45}}{D_{35}}.$

Hence, $\ {}_{10:20}V_{25} =$

and $1000\ {}_{10:20}V_{25} =$

2. Find the terminal reserve of the sixth policy year on a 15-payment life policy of \$3000 taken at age 30. *Ans.* \$452.16.

3. Find the terminal reserve of the twentieth policy year on a 20-payment life policy for \$1000 issued at age 25. Explain why this result equals the net single premium at age 45.

125. Reserves for endowment insurance. For an r–year endowment insurance with premiums payable for m years, the nth year terminal reserve is denoted by

$$_{n\,m}V_{x\overline{r|}}.$$

From the principle that the nth year terminal reserve is equal to the net premium for the benefit at age $x + n$ less the present value of future premiums, we have

$$_{n:m}V_{x\overline{r|}} = A_{x+n:\overline{r-n|}} - {}_{m}P_{x\overline{r|}}\,(1 + a_{x+n:\overline{m-n-1|}}). \tag{6}$$

For the very common case in which the annual premiums are payable for the entire endowment period, we have $m = r$.

Then from formula (6), or from the principle stated above, it follows that

$$ {}_nV_{x\overline{r}|} = A_{x+n:\overline{r-n}|} - P_{x\overline{r}|}\,(1 + a_{x+n:\overline{r-n-1}|}). \tag{7}$$

Formula (7) may be changed into another useful form by the substitution

$$A_{x+n:\overline{r-n}|} = P_{x+n:\overline{r-n}|}\,(1 + a_{x+n:\overline{r-n-1}|}).$$

This gives

$$ {}_nV_{x\overline{r}|} = (P_{x+n:\overline{r-n}|} - P_{x\overline{r}|})\,(1 + a_{x+n:\overline{r-n-1}|}), \tag{8}$$

which is analogous to (2) and (5).

EXERCISES

1. Give a verbal interpretation to formula (8) similar to those given to formula (2) Art. 123, and formula (5), Art. 124.

2. Find the terminal reserve of the tenth policy year on a twenty-year endowment policy of $1000 taken at age 25. *Ans.* $396.21.

3. Write formulas similar to formulas (7) and (8), but for term insurance for a term of r years.

4. Find the fifth year terminal reserve on a ten-year term policy of $1000 issued at age 22. *Ans.* $0.82.

126. Retrospective method of computing reserves. The methods of computing reserves thus far discussed are known as prospective methods because they look to the future from a given insurance situation. Other methods known as **retrospective methods** consist in accumulating the net premiums received at the assumed rate of interest and deducting the death losses.

To describe one such method, we show first how to obtain from the reserve of any policy year that of the following policy year. To this end, let ${}_nV_x$ be the terminal reserves of the nth year on an insurance of 1, and P_x, the net annual premium.

Then ${}_nV_x + P_x$ is the reserve at the beginning of the $(n+1)$th year, called the **initial reserve** of the $(n+1)$th year.

Hence, $l_{x+n}({}_nV_x + P_x)$ is the aggregate reserve at the beginning of $(n+1)$th year for the l_{x+n} persons insured.

If this amount be improved at interest, the accumulation at the end of the year will be

$$l_{x+n}({}_nV_x + P_x)(1+i).$$

During the $(n+1)$th year, d_{x+n} persons will have died and the company will be called upon to pay d_{x+n} in death claims. The amount left after the payment of these death claims will then be

$$l_{x+n}({}_nV_x + P_x)(1+i) - d_{x+n},$$

which represents the aggregate policy reserves belonging to the l_{x+n+1} survivors.

We may therefore write

$$l_{x+n+1}\ {}_{n+1}V_x = l_{x+n}\ ({}_nV_x + P_x)(1+i) - d_{x+n}.$$

Hence,

$$\begin{aligned}
{}_{n+1}V_x &= \frac{l_{x+n}({}_nV_x + P_x)(1+i)}{l_{x+n+1}} - \frac{d_{x+n}}{l_{x+n+1}}, \\
&= \frac{l_{x+n}}{vl_{x+n+1}}({}_nV_x + P_x) - \frac{d_{x+n}}{l_{x+n+1}}, \\
&= \frac{v^{x+n}l_{x+n}}{v^{x+n+1}l_{x+n+1}}({}_nV_x + P_x) - \frac{v^{x+n+1}d_{x+n}}{v^{x+n+1}l_{x+n+1}}, \\
&= \frac{D_{x+n}}{D_{x+n+1}}({}_nV_x + P_x) - \frac{C_{x+n}}{D_{x+n+1}},
\end{aligned}$$

If now we define the valuation factors (Table XI)

$$u_x = \frac{D_x}{D_{x+1}} \text{ and } k_x = \frac{C_x}{D_{x+1}},$$

we have

$${}_{n+1}V_x = u_{x+n}({}_nV_x + P_x) - k_{x+n}. \qquad (9)$$

This formula is known as **Fackler's Accumulation Formula.** The u_{x+n} and k_{x+n} are independent of the form of the policy. For a varying amount of insurance from year to year, the factor multiplying k_x would, of course, change.

In particular, it should be kept in mind in finding the reserve of the first policy year by this method that

$$n = 0 \text{ and } {}_0V_x = 0 \text{ in (9)},$$

and hence to find the reserve for the first year we simply accumulate the first net premium P_x and subtract k_x. The Fackler formula is perhaps used by actuaries more than any other formula when preparing complete tables of policy reserves. Its chief advantage lies in the fact that the reserve calculation at the end of any year is made to depend upon that of the previous year; and, thus by checking the calculations of the reserve for a given policy year, we have an automatic check for the lower policy years.

A table of valuation functions u_x and k_x according to the American Experience Table of Mortality and 3½ per cent interest is given as Table XI.

The student should verify a number of these values by means of the C_x and D_x functions. It should, however, be stated that the u_x and k_x are given to more significant figures than can be obtained accurately from the C and D columns of Table X.

EXERCISES AND PROBLEMS

1. By Fackler's accumulation formula, find the terminal reserves for each of the first ten policy years on a $1000 ordinary life policy taken at age 25. Check your result for the tenth year by using the prospective method.

FORM FOR SOLUTION: In this case,

$$P_{25} = 0.0151034.$$

For the first year, ${}_1V_{25} = u_{25}P_{25} - k_{25} =$

For the second year, ${}_2V_{25} = u_{26}\,({}_1V_{25} + P_{25}) - k_{26} =$

. . . .

For the tenth year, ${}_{10}V_{25} = u_{34}\,({}_9V_{25} + P_{25}) - k_{34} =$

CHECK: By the prospective formula for the tenth policy year,

$${}_{10}V_{25} = A_{35} - P_{25}(1 + a_{35}) =$$

2. A man now aged 45 has a twenty year endowment policy of $1000 taken at age 30 and on which the terminal reserve is $664.91 at the end of

the fifteenth policy year. What will be the terminal reserve of the sixteenth policy year? *Ans.* $726.02.

3. In problem 2, calculate the terminal reserves for succeeding policy years until the policy matures.

4. Find the terminal reserves for each of the first ten policy years on a twenty-payment life policy for $1000 taken at age 30. Check your result by finding the terminal reserve of the tenth year by the prospective method.

127. Gross premiums. Premiums heretofore discussed have been net premiums or the mathematical equivalent of the benefits. Since the insurance business, like any other business, cannot be conducted without expense, it is necessary that the companies add to the net premiums certain amounts called **loadings** with which to meet the expenses incident to the business and also to provide against unforseen contingencies. The net premium plus the loading is called the **gross premium.** Sometimes the loading is a percentage of the net premium, uniform at all ages, sometimes a percentage of the net premium varying with the age, and again, it may be a percentage of the net premium plus a constant.

It is not within the scope of this book to discuss in detail the theory of loadings. Suffice it to say that the loading formulas used by different companies vary greatly but after all produce very similar gross premiums for the same kinds of policies.

128. Preliminary term valuation. In our discussion of net premiums and reserves we have possibly carried along the general implication that if the gross premium under a policy were the same for all years, the net premium and consequently the loading would also be the same for all years. In other words, we have perhaps implied that level gross premiums have corresponding level net premiums. In our discussion of reserves we have also tacitly assumed level net premiums.

Since in the first year of a policy a life insurance company must pay a relatively large agent's commission and also medical and inspection fees, it is clear that the expense for the first policy year is heavier than that for subsequent policy years. However, under the level net premium method the loading is

exactly the same in the first year and subsequent years. There thus results a deficit in the business in its first policy year. This deficit must be met from the general surplus funds of the company. In the case of a company with small surplus it is practically impossible to operate and build up an adequate surplus under this method. Various methods have been devised to give relief to these conditions. One of these methods, known as the **preliminary term** system, will be described.

Under the preliminary term method it is assumed that all of the first year premium is available for current mortality and expense. That is, the first year's insurance is considered as term insurance, the policy providing that it may be renewed at the end of the first year as a life or endowment policy at the same gross premium. This therefore results in a first year net premium equal to the natural premium for the age at issue. The excess of the gross premium over the natural premium for the first year therefore becomes first year loading and is available for first year expenses. The net premium for subsequent years is the net premium for the benefits at an age one year in advance of the age at issue.

For example, if an ordinary life policy is issued at age twenty-five, under the preliminary term system at a gross annual premium of $19.50, the first year net premium according to the American Experience Table of Mortality and 3½ per cent interest would be $7.79, thus leaving $11.71 as first year loading. The net premium for subsequent years would be the ordinary life level net premium for age twenty-six, which according to the American Experience Table of Mortality and 3½ per cent interest is $15.48. This results in a renewal loading of $4.02. Had the policy been issued under the level net premium system, the net premium for all years would have been $15.10 resulting in a uniform loading of $4.40.

Again, a twenty-payment life policy issued at age twenty-five, under the preliminary term system at a gross annual premium of $28.75 would also have a first year net premium of $7.79, thus releasing $20.96 for first year expenses. The net premium for the subsequent nineteen years would be the level

net premium for a nineteen payment life policy at age twenty-six. This according to the American Experience Table of Mortality and $3\frac{1}{2}$ per cent interest is $23.63 and the loading during these years would be $5.12 a year. Under the level net system of valuation the net premium of all years would be $22.53 and the corresponding loading would be $6.22 per annum.

It will thus be seen that the preliminary term system in connection with these forms of insurance really resolves itself into a redistribution of loading roughly corresponding to the incidence of the expense.

It should be evident to the student that, since under this system the entire gross premium is available for first year mortality and expense, there is no terminal reserve at the end of the first policy year. The accumulation of the reserve begins in the second year and necessarily must be less than the level net premium reserve until the policy becomes paid up.

The preliminary term system of valuation when applied to ordinary life policies and to limited payment life and endowment policies with long premium paying periods is recognized as sound in principle by the best authorities. It is, however, objectionable when applied to limited payment life or endowment policies with short premium paying periods. This objection can be best explained by means of an illustration.

Assume that a ten-year endowment insurance policy for $1,000 were issued at age twenty-five at a gross annual premium of $101.75. The level net premium is $86.45. The net first year premium under the preliminary term system would again be $7.79, thus resulting in a first year loading of $93.96. This is far more than is necessary for first year expenses. It is therefore necessary to introduce some modification of the preliminary term system when applied to policies with short premium paying periods. Several modifications of the preliminary term system, designed to overcome the objections explained above, have been devised and with such modifications the preliminary term system of valuation has been adopted as the standard of valuation by the legislatures of most of the states.

Another system of valuation designed to accomplish the same purpose as the preliminary term system is the **select and ultimate** which has been adopted as a standard in the State of New York. It is beyond the scope of this book to explain this system or the various modifications of the preliminary term system.

The student who is interested in the details of the various standards of valuation may well consult:

Moir's *Life Assurance Primer,*

Fackler's *Notes on Life Insurance,*

Dawson's *Comparative Reserve Tables,*

The Record of The American Institute of Actuaries, Vol. VIII, Part II, pp. 314–360.

In bringing to a conclusion the foregoing three chapters on the mathematics of life insurance, it should be emphasized that we have merely touched on a broad field. We have confined our treatment to a few fundamental topics. No mention has been made of the distribution of surplus, of joint life insurance and annuities, of survivorship insurance and annuities, of the preparation of annual statements including the gain and loss exhibit required by state departments of insurance, nor of many other topics that might well be considered if space permitted. It is hoped that in the short space available, we have given a general view of the methods of approach to the quantitative treatment of life insurance, and that the student has been able to form from this treatment an appreciation of the beautiful system of long time finance that underlies legal reserve life insurance.

The next step of the interested student in going forward to study the mathematics of insurance without the guidance of a teacher of actuarial science, may well consist of a study of portions of the Text-Book of the Institute of Actuaries, and in following the courses of study outlined by the Educational Committees of the Actuarial Society of America and of the American Institute of Actuaries.

MISCELLANEOUS EXERCISES AND PROBLEMS

1. Find the level net terminal reserves of the first, second, and third policy years on an ordinary life policy of $1000 taken at age 21. *Ans.* $6.45, $13.13, and $20.04.

2. Same as problem 1 except that the policy is a 20 payment life policy. *Ans.* $14.05, $28.65, and $43.84.

3. Same as problem 1, except that the policy is a 20-year endowment. *Ans.* $32.71, $66.78, and $102.28.

4. Find the level net terminal reserve of the fifth year on a 20-year endowment insurance for $1000 taken at age 21, with premiums payable for 10 years.

5. Find the level net terminal reserve of the nineteenth year on a 20-payment life policy for $8000 taken at age 23. *Ans.* $3246.88.

6. Same as problem 5, but for a 20-year endowment policy. *Ans.* $7417.20.

7. Under the preliminary term system of valuation, find the terminal reserve for the tenth policy year on an ordinary life policy for $1000 taken at age 25.

Form for Solution: Since the insurance of the first year is term insurance, we need simply to find the value of

$$ {}_9V_{26} = A_{35} - P_{26}(1 + a_{35}), $$
$$ = $$

8. For an ordinary life policy of $1000 taken at age 25, find the terminal reserves of each of the first ten policy years under the preliminary term system of valuation, using Fackler's Accumulation Formula.

Hint: The value to be accumulated for the second policy year is simply the ordinary net annual premium at age 26.

9. For a twenty-payment life policy of $1000, taken at age 30, find the terminal reserve of the 19th policy year both under the level net premium system and under the preliminary term system of valuation. Compare the resulting reserves. *Ans.* Under level net premium, $472.81; under preliminary term, $471.49; difference, $1.32.

10. A twenty-year endowment policy of 1 issued at age x is to be valued under the following modified preliminary term system known as the modification to ordinary life basis:

The net premium of the first year is to be that for one year term insurance plus a certain excess e. The subsequent net annual premiums are the level

net ordinary life premiums for age $x+1$, plus this same excess e required to mature the contract.

Find, under this modification, the terminal reserves of the first five policy years on a 20-year endowment policy of \$1000 issued at age 25.

Outline of Solution: To find e, we bear in mind that, apart from the accumulations from this excess as a pure endowment, the reserve at the end of 20 years would be only the level net premium terminal reserve of the 19th policy year on an ordinary life policy of 1 issued at age 26, or ${}_{19}V_{26}$.

But at the end of the 20th year, the value of a 20-year endowment policy of 1 is 1.

Hence, the e per year must provide at maturity date a pure endowment of

$$1 - {}_{19}V_{26}.$$

The e is the annual payment on a forborne temporary annuity due at age 25 (Art. 111), that will accumulate in 20 years to a sum

$$1 - {}_{19}V_{26}.$$

Hence, by Art. 111,

$$e\,\frac{N_{26} - N_{45}}{D_{45}} = 1 - {}_{19}V_{26},$$

or

$$e = (1 - {}_{19}V_{26})\frac{D_{45}}{N_{25} - N_{45}} =$$

The net premium for the first year is \$1000 $(P^{\,1}_{25\,\overline{1|}} + e)$.

The net premium for any subsequent year is \$1000 $(P_{26} + e)$.

Then by the Fackler Accumulation formula (Art. 126), the terminal reserve of the first year is given by \$1000 $u_{25}e =$

The terminal reserve of the second year is given by \$1000 [u_{26} (initial reserve of second year) — k_{26}] =

It is left as an exercise for the student to complete the solution. *Ans.* First year, \$25.28; the second year, \$59.61; third year, \$95.37; fourth year, \$132.64; fifth year, \$171.46.

CHAPTER XI

LOGARITHMS

129. Uses of logarithms. Computations that involve extensive multiplications and divisions or the finding of powers and roots are greatly facilitated by the use of logarithms.

When, for example, it is necessary to compute the amount of \$100 for a period of 40 years at 6 per cent convertible annually, the work of computing the value of

$$100(1.06)^{40}$$

by multiplying together 40 factors each equal to 1.06 would be very laborious. Logarithms afford a short method of making the computation. As another illustration, suppose that money invested has earned interest until \$1000 amounts to \$2000 in 10 years. What is the rate of interest convertible annually? In this case, let x be the rate. Then

$$1000\,(1 + x)^{10} = 2000,$$

$$(1 + x)^{10} = 2,$$

$$1 + x = \sqrt[10]{2},$$

and

$$x = \sqrt[10]{2} - 1.$$

The difficult process of extracting the 10th root can, by the use of logarithms, be replaced by a simple process of division. Another important use of logarithms arises in the attempt to solve certain important equations with the unknown in the exponent. For an example, see (12) Art. 9.

To understand the uses of logarithms, a study of the definition and underlying principles upon which logarithms are based is desirable.

130. Definition of a logarithm. *If* $a^x = y (a > 0,\ a \neq 1)$, *then* x *is said to be the* **logarithm** *of* y *to the base* a, and this relation is written

$$x = \log_a y.$$

The two equations
$$a^x = y \tag{1}$$
and
$$x = \log_a y \tag{2}$$

thus mean exactly the same thing; and the terms *logarithm and exponent are equivalent.*

We shall assume in what follows that:

1. *Corresponding to any two positive numbers* y *and* a $(a \neq 1)$ *there exists one and only one real number* x *such that* $a^x = y$.

This assumption is sometimes expressed by saying that any positive number has one and only one logarithm, whatever positive number is the base (unity excepted).

2. *The laws of exponents which apply to rational exponents are also valid when irrational exponents are involved.*

Thus, $a^{\sqrt{2}} \cdot a^{\sqrt{2}} = a^{2\sqrt{2}}$.

EXERCISES

1. $\log_2 8 = ?$ $\log_a a = ?$ $\log_4 2 = ?$ $\log_4 256 = ?$ $\log_{16} 4 = ?$

2. Fill out the following table.

BASE	NUMBER	LOGARITHM
	81	4
10		5
	23	$\frac{1}{3}$
2	$\frac{1}{32}$	
3	$\frac{1}{27}$	
	32	-5

131. Derived properties of logarithms.

1. *The logarithm of a product equals the sum of the logarithms of its factors.*

Let $\log_a u = x$ and $\log_a v = y$, (1)

then, $a^x = u,\ a^y = v,$ (Definition of logarithm.)

and $uv = a^{x+y}.$ (Art. 130, Assumption 2.)

Hence, $\log_a uv = x + y,$

that is, $\boldsymbol{\log_a uv = \log_a u + \log_a v.}$

Similarly, $\boldsymbol{\log_a(uvw) = \log_a u + \log_a v + \log_a w,}$

and so on for any number of factors.

EXAMPLE: $\log_{10} 255 = \log_{10} 3 + \log_{10} 5 + \log_{10} 17.$

2. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

As above, let $\log_a u = x$ and $\log_a v = y,$

then, $a^x = u,\ a^y = v,$

and $\dfrac{u}{v} = a^{x-y}.$

Hence, $\log_a \dfrac{u}{v} = x - y,$

that is, $\boldsymbol{\log_a \dfrac{u}{v} = \log_a u - \log_a v.}$

EXAMPLE: $\log_{10} \frac{625}{133} = \log_{10} 625 - \log_{10} 133.$

3. *The logarithm of u^v is equal to v times the logarithm of u.*

To prove this, let $x = \log_a u$ or $a^x = u.$ (1)

Then, $u^v = a^{vx}.$ (Law of Indices, and Art. 130, 2.)

Hence, $\boldsymbol{\log_a u^v = vx = v \log_a u.}$ (2)

EXAMPLE: $\log_{10} (257)^{\frac{1}{2}} = \frac{1}{2} \log_{10} 257.$

Making $v = n$ and $v = \frac{1}{n}$ respectively, we have

(*a*) *The logarithm of the nth power of a number is n times the logarithm of the number.*

(*b*) *The logarithm of the real positive nth root of a number is the logarithm of the number divided by n.*

EXERCISES

Express the logarithms of the following expressions in terms of the logarithms of integers.

1. * $\log \dfrac{\sqrt[4]{8}}{9^{\frac{1}{5}}6^{\frac{2}{3}}}$.

SOLUTION: $\log \dfrac{\sqrt[4]{8}}{9^{\frac{1}{5}}6^{\frac{2}{3}}} = \log \sqrt[4]{8} - \log 9^{\frac{1}{5}} - \log 6^{\frac{2}{3}}$, (1 and 2, Art. 131.)

$= \frac{1}{4}\log 8 - \frac{1}{5}\log 9 - \frac{2}{3}\log 6.$ (3, Art. 131.)

2. $\log \dfrac{2^2}{3^3}$. **3.** $\log \dfrac{\sqrt{13}}{\sqrt[5]{10}\,\sqrt[3]{48}}$. **4.** $\log \dfrac{25^{\frac{1}{3}}}{11^{\frac{2}{3}}23^{\frac{3}{4}}}$.

Express the logarithms of the following in terms of the logarithms of prime numbers.

5. $\log \dfrac{(35)^{\frac{1}{2}}}{(36)^2(72)^{\frac{1}{4}}}$. **6.** $\log \dfrac{(88)^{-\frac{1}{2}}}{(75)^{\frac{3}{4}}(12)^2}$.

7. $\log \dfrac{(100)^2}{(20)^{\frac{1}{2}}(75)^{\frac{2}{3}}}$. **8.** $\log (\sqrt{2}\sqrt[3]{7^2}\sqrt[5]{6})$.

9. $\log (\sqrt[3]{2}\sqrt{66}\,\sqrt[5]{25})$. **10.** Prove that $\log_a 1 = 0$.

Given $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, $\log_{10} 7 = 0.8451$, find the logarithms of the following numbers to the base 10.

11. 12. **12.** 30. **13.** 42. **14.** 420.

15. 189. **16.** 900. **17.** 343. **18.** $\frac{343}{16}$.

* If the same base is used throughout in a problem, it is customary not to write the base.

19. $\frac{35}{48}$. **20.** $\frac{1}{252}$. **21.** $\frac{1}{1029}$. **22.** $\sqrt{504}$.

23. $\sqrt[3]{294}$. **24.** $\sqrt{\frac{1}{2}}$. **25.** $\sqrt[5]{1029}$. **26.** $\sqrt[6]{43218}$.

132. Common logarithms. While any positive number can be used as the base of some system of logarithms, there are two systems in general use. These are the **common or Briggs's system** and the **natural or Naperian system.** In the common system the base is 10, while in the natural system the base is a certain irrational number $e = 2.71828 \ldots$ It may be stated that the common system is adapted to numerical computation, while the natural system is adapted to analytical work.*

In the following discussion of common logarithms, $\log x$ is written as an abbreviation of $\log_{10} x$.

Since

$$\begin{array}{ll} 10^0 = 1 & 10^{-1} = 0.1 \\ 10^1 = 10 & 10^{-2} = 0.01 \\ 10^2 = 100 & 10^{-3} = 0.001 \\ 10^3 = 1000 & 10^{-4} = 0.0001 \\ \cdots & \cdots \end{array}$$

it follows that

$$\begin{array}{ll} \log 1 = 0 & \log 0.1 = -1 \\ \log 10 = 1 & \log 0.01 = -2 \\ \log 100 = 2 & \log 0.001 = -3 \\ \log 1000 = 3 & \log 0.0001 = -4 \\ \cdots & \cdots \end{array}$$

So far as these powers of 10 are concerned, it may be observed that the logarithm of the number becomes greater as the number increases. In accordance with this observation, we may assume, if $a < x < b$, that

$$\log a < \log x < \log b. \tag{1}$$

For example, $\log 100 < \log 765 < \log 1000$,

or $2 < \log 765 < 3$.

* The notation $\ln x$ for $\log_e x$ and $\log x$ for $\log_{10} x$ is frequently used when both kinds of logarithms appear in the same problem.

When the logarithm of a number is not an integer, it may be represented at least approximately by means of decimal fractions. Thus, $\log 765 = 2.8837$ correct to four decimal places.

The integral part of a logarithm is called the **characteristic** and the decimal part is called the **mantissa.** In log 765, the characteristic is 2 and the mantissa is 0.8837. For convenience in constructing tables, it is desirable to select the mantissa as positive even if the logarithm is a negative number. For example, $\log \frac{1}{2} = -0.3010$; but since $-0.3010 = 9.6990 - 10$, this may be written $\log \frac{1}{2} = 9.6990 - 10$ with a positive mantissa. The following illustration shows the method of writing the characteristic and mantissa:

$$\begin{aligned}
\log 7185 &= 3.8564\\
\log 718.5 &= 2.8564\\
\log 71.85 &= 1.8564\\
\log 7.185 &= 0.8564\\
\log 0.7185 &= 9.8564 - 10\\
\log 0.07185 &= 8.8564 - 10.
\end{aligned}$$

133. Characteristic. With our decimal system of notation, the characteristic in the case of the base 10 is very easy to determine by a simple rule. Herein lies the advantage of this base.

If y is a number which has n digits in the integral part, then

$$10^{n-1} \leqq y < 10^n, \qquad (1)$$

and by Art. 132, (1), $n - 1 \leqq \log y < n$.

Hence, $$\log y = n - 1 + f,$$

where f is positive and less than 1.

Rule I. *To find the characteristic of the common logarithm of a number which has an integral part, subtract 1 from the number of digits in the integral part.*

If y represents a decimal fraction, we may move the decimal point ten places to the right, and apply the rule just stated to

the integral part of the number so formed, provided we subtract 10 from the resulting logarithm. That is,

$$y = \frac{10^{10}\,y}{10^{10}} = \frac{y_1}{10^{10}},$$

where
$$\begin{aligned}\log y &= \log y_1 - \log 10^{10},\\ &= \log y_1 - 10.\end{aligned}$$

The result so obtained could manifestly also be obtained by the following:

RULE II. *To find the characteristic of the common logarithm of a decimal fraction, subtract from 9 the number of ciphers between the decimal point and the first significant figure. From the number so obtained subtract 10.*

If two numbers contain the same sequence of figures, and therefore differ only in the position of the decimal point, the one number is the product of the other and an integral power of 10, and hence, by Art. 131, the logarithms of the numbers differ only by an integer. Thus,

$$\begin{aligned}\log 3722 &= \log 37.22 + \log 100,\\ &= \log 37.22 + 2.\end{aligned}$$

Hence, *the mantissa of the common logarithm of a number is independent of the position of the decimal point.* In other words, the common logarithms of two numbers which contain the same sequence of figures differ only in their characteristics. Hence, tables of logarithms contain only the mantissas, and the computer must find the characteristics by the foregoing rules.

134. Use of tables. On pp. 226, 227, a four-place table of logarithms is given. In this table, the mantissas of the logarithms of all integers from 1 to 999 are recorded correct to four decimal places. Five-place, six-place, and seven-place tables are in common use, but this four-place table will serve to illustrate the method of using logarithms in simple calculations. We

give some problems in this book that require a seven-place table for their solution.

Methods by which a table of logarithms can be made will be discussed in Arts. 144 and 145 after applying the logarithms found in the table to purposes of arithmetical calculation. In order to use the tables we must know how to take from the tables the logarithm of a given number, and how to take from the tables the number which has a given logarithm.

135. To find from the table the logarithm of a given number.

EXAMPLES

1. Find the logarithm of 821.

Glance down the column headed N for the first two significant figures, then at the top of the table for the third figure. In the row with 82 and the column with 1 is found 9143.

Hence, $$\log 821 = 2.9143.$$

2. Find the logarithm of 68.42.

This number has more than three significant figures, so that its logarithm is not recorded in the table. It may, however, be obtained approximately from logarithms recorded in the table by a process of **interpolation.** In this process, it is assumed that to a small change in the number, there corresponds a change in the logarithm which is proportional to the change in the number. This assumption is called the principle of proportional parts. As in example 1, we find that the mantissas of 6840 and 6850 are 8351 and 8357, respectively. The difference between these two mantissas is 6. Since 6842 is two-tenths of the interval from 6840 to 6850, by the principle of porportional parts, we add to 8351,

$$0.2 \times 6 = 1^{+}.$$

Hence, $$\log 68.42 = 1.8352.$$

136. To find from the table the number which corresponds to a given logarithm.

EXAMPLES

1. Find the number whose logarithm is 2.4675.

The mantissa 4675 is not recorded in the table, but it lies between the two adjacent mantissas 4669 and 4683 of the table. The mantissa 4669 corresponds to the number 293 and 4683 corresponds to 294. The number

1139
.569

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	6459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	9854	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

675 is $\frac{6}{14}$ of the interval from 4669 to 4683. By the principle of proportional parts, the number whose mantissa is 4675 is

$$2930 + \tfrac{6}{14} \times 10 = 2934^{+}.$$

Hence, $\log 293.4 = 2.4675.$

2. Find the number whose logarithm is $9.3025 - 10$.

From the table,

$$\begin{aligned} \log 0.2000 &= 9.3010 - 10 \\ \log 0.2010 &= \underline{9.3032 - 10} \\ \text{Difference} &= 0.0022 \end{aligned}$$

$$(9.3025 - 10) - (9.3010 - 10) = 0.0015.$$

By the principle of proportional parts, the number is

$$0.2000 + \tfrac{15}{22} \times 0.0010 = 0.2007.$$

EXERCISES

Obtain, from the table, the common logarithms of the following:

1. 163. **2.** 89. **3.** 999.
4. 1.41. **5.** 0.00785. **6.** 6563.
7. 7.854. **8.** 3.142. **9.** 0.5236.
10. 1.732. **11.** 0.8665. **12.** 0.0298.

Obtain, by means of the table, the numbers whose common logarithms are the following:

13. 2.7182. **14.** $9.8532 - 10$. **15.** 3.1416.
16. 0.5236. **17.** $7.8321 - 10$. **18.** 4.2631.
19. $8.5432 - 10$. **20.** 1.4142. **21.** 0.4343.

137. Computation by means of logarithms. The application of logarithms to shorten calculations depends upon the properties of logarithms given in Art. 131. By means of logarithms laborious multiplications and divisions may be replaced by additions and subtractions; and involution and evolution may be replaced by multiplication and division.

EXAMPLES

1. Find the value of $N = \dfrac{6.320 \times 8.674}{2.851}$ to four significant figures.

$$\log 6.320 = 0.8007$$
$$\log 8.674 = 0.9382$$

$$\begin{aligned} \log (6.320)(8.674) &= \overline{1.7389} \\ \log 2.851 &= 0.4550 \\ \log N &= 1.2839 \\ N &= 19.23. \end{aligned}$$

In using logarithms, *much time is saved and the liability of error is decreased by making a so-called form for all the work before using the table at all.*

Thus, in example 1, the form is

$$\begin{aligned} \log 6.320 &= \\ \log 8.674 &= \\ \log (6.320)(8.674) &= \\ \log 2.851 &= \\ \log N &= \\ N &= \end{aligned}$$

2. Make a form for evaluating $N = \dfrac{(6.85)^{\frac{1}{2}} \sqrt[3]{8.542}}{\sqrt{65.27}}$.

$$\begin{aligned} \log 6.85 &= \\ \log (6.85)^{\frac{1}{2}} &= \\ \log 8.542 &= \\ \log \sqrt[3]{8.542} &= \\ \log (6.85)^{\frac{1}{2}} \sqrt[3]{8.542} &= \\ \log 65.27 &= \\ \log \sqrt{65.27} &= \\ \log N &= \\ N &= \end{aligned}$$

3. Evaluate $N = \sqrt[3]{-58.61}$.*

$$\begin{aligned} \log 58.61 &= 1.7680\, n \\ \log \sqrt[3]{58.61} &= 0.5893\, n \\ N &= -3.885. \end{aligned}$$

* When a number is negative, find its logarithm without regard to sign, writing n after a logarithm that corresponds to a negative number so as to keep the negative sign in mind.

EXERCISES AND PROBLEMS

Evaluate to four significant figures by logarithms.

1. $\dfrac{2550 \cdot 317.3}{731}$. *Ans.** $= 1107$.

2. $(0.9314)^4$. *Ans.* $= 0.7523$.

3. $\sqrt[3]{753}$. *Ans.* $= 9.098$.

4. $\dfrac{\sqrt{5609}}{(1.05)^3}$. *Ans.* $= 64.69$.

5. $(3.456)^8$. *Ans.* $= 20360$.

6. $(0.9093)^{-4}$. *Ans.* $= 1.463$.

7. $725\,(1.06)^{10}$. *Ans.* $= 1298$.

8. $1800\,(1.035)^{25}$. *Ans.* $= 4244$.

9. $2625\,(1.08)^{15}$.

10. $\dfrac{1254\,(1.045)^{\frac{1}{2}}}{941}$. *Ans.* $= 1.362$.

11. $\dfrac{96\,(1.065)^{\frac{1}{3}}}{1069}$.

12. $\dfrac{426\,(1.07)^{\frac{1}{4}}}{0.07}$. *Ans.* $= 6190$.

13. $\dfrac{5410\,(1.0375)^{\frac{3}{4}}}{0.0375}$.

14. $9000\,(1.085)^{\frac{1}{12}}$. *Ans.* $= 9062$.

15. $\dfrac{8375\,(1.0425)^{\frac{7}{12}}}{8300}$.

16. $\dfrac{845\,(1.04)^{-5}}{0.04}$. *Ans.* $= 17370$.

17. $(1.055)^{-10}$. *Ans.* = 0.58[illegible]1.

18. $\dfrac{(1.025)^{-10}}{(1.035)^{-10}}$.

* Use four-place table in these exercises except when there is a statement to the contrary. Take the nearest even number rather than the nearest odd number when the neglected part is a 5 in the first place not retained and the figures following are zeros or unknown.

19. $(1.09)^{-\frac{1}{2}}$. *Ans.* $= 0.9578$.

20. $(1.075)^{-\frac{3}{2}}$.

21. $\dfrac{(1.035)^9 - 1}{0.035}$. *Ans.* $= 10.4$.*

22. $\dfrac{(1.045)^7 - 1}{0.045}$.

23. $\dfrac{1 - (1.04)^{-6}}{0.04}$. *Ans.* $= 5.24$.

24. $\dfrac{1 - (1.045)^{-6}}{0.045}$.

25. $\dfrac{0.0425}{(1.0425)^8 - 1}$. *Ans.* $= 0.108$.

26. $\dfrac{0.07}{(1.07)^{11} - 1}$.

27. $\dfrac{0.065}{1 - (1.065)^{-7}}$. *Ans.* $= 0.183$.

28. $\dfrac{(1.075)^3 - 1}{(1.075)^4 - 1}$.

29. $10000 \cdot \dfrac{0.055}{0.045} \cdot \dfrac{1 - (1.045)^{-10}}{1 - (1.055)^{-10}}$. *Ans.* $= 10500$.

30. $5000 \cdot \dfrac{0.04}{0.0375} \cdot \dfrac{1 - (1.0375)^{-5}}{1 - (1.04)^{-5}}$.

31. $\dfrac{(1.06)^{\frac{79}{12}} - 1}{(1.06)^{\frac{1}{12}} - 1}$. *Ans.* $= 96.06$ with a seven-place table.

Use a seven-place table of logarithms. Try also a four-place table and compare.

32. $\dfrac{1 - (1.073)^{\frac{77}{12}}}{1 - (1.075)^{\frac{1}{12}}}$.

Use a seven-place table of logarithms. Try also four-place table and compare.

33. If \$1500 is placed at 4 per cent interest converted annually, what will it amount to in 8 years?

* In such a computation as is involved here, we often lose one or more significant figures due to a subtraction. This may reduce the number of significant figures that we record in the answer. Exercises 22–30 illustrate this point.

HINT: In n years \$1 will amount to $(\$1.04)^n$. To find the answer to the nearest cent it will be necessary to use a six or seven-place logarithmic table.

34. What will \$12,000 amount to if left at interest for 10 years at $4\frac{1}{4}$ per cent: (*a*) converted annually? (*b*) converted semiannually? (*c*) converted quarterly? Obtain result correct to a dime.

35. What sum of money left for 20 years at 5 per cent, converted annually, will amount to \$10,000 at the end of that time?

36. If \$1 had been kept at interest 5 per cent, compounded annually, from the beginning of the Christian era to the present time, how many digits would occur in the integral part of the accumulated amount when expressed in dollars? *Ans.* 41 digits until the year 1935, when it will require 42.

37. The formula $y = ks^x g^{c^x}$, where $\log k = 5.03370116$, $\log s = -0.003296862$, $\log g = -0.00013205$, $\log c = 0.04579609$, gives the number living at age x in Hunter's Makehamized American Experience Table of Mortality. Find, to such a degree of accuracy as you can secure with a four-place table of logarithms, the number living (1) at age 10, (2) at age 30.

138. Change of base.

The logarithm of a number y to the base b is equal to the product of its logarithm to the base a and the logarithm of a to the base b, that is

$$\log_b y = \log_a y \,.\, \log_b a. \tag{1}$$

Let
$$u = \log_a y \text{ and } v = \log_b y. \tag{2}$$

Then,
$$a^u = y,\ b^v = y, \tag{3}$$

and
$$a^u = b^v. \tag{4}$$

$$a = b^{\frac{v}{u}}, \tag{5}$$

$$\frac{v}{u} = \log_b a,$$

$$v = u \log_b a. \tag{6}$$

From (2) and (6)
$$\boldsymbol{\log_b y = \log_a y \log_b a.} \tag{7}$$

EXAMPLE: $\log_{10} 128 = \log_2 128 \log_{10} 2.$

By making $y = b$ in (7), we obtain

$$1 = \log_a b \log_b a.$$

That is,
$$\log_b a = \frac{1}{\log_a b}. \tag{8}$$

The number $\log_b a$ is often called the **modulus** of the system of base b with respect to the system of base a.

In Art. 133, attention is called to the advantages of 10 for the base of a system of logarithms to be used in numerical calculations. For analytical purposes, as will appear in the calculus, it is convenient to use natural logarithms. This system has for its base an irrational number $e = 2.7182818 \ldots$ In Art. 141, there is given a series from which this approximation to e is obtained, and in Art. 144 another series from which the logarithm of a number to the base e can be obtained to any number of decimal places. It turns out that

$$\log_e 10 = 2.3025851,$$

and
$$\log_{10} e = \frac{1}{\log_e 10} = 0.4342945.$$

By (1),
$$\log_{10} y = \log_e y \log_{10} e,$$
$$= 0.4342945 \log_e y,$$

or
$$\log_e y = 2.3025851 \log_{10} y.$$

The number $\log_{10} e = 0.4342945$ is the **modulus** (to seven significant figures) of common logarithms with respect to natural logarithms.

EXERCISES

1. Given $\log_e 2 = 0.6931$, find $\log_{10} 2$ and compare the result with the value in table, p. 226.

2. Given $\log_{10} 2 = 0.3010$, find $\log_e 4$.

3. Given $\log_{10} 3$, find $\log_5 3$.

HINT: By Art. 138,
$$\log_5 3 = \log_{10} 3 \cdot \log_5 10,$$
$$= \frac{\log_{10} 3}{\log_{10} 5}.$$

4. Given $\log_{10} 3$, find $\log_5 81$.

Find the logarithms of the following numbers:

5. 10 to the base 2.
6. 10 to the base 4.
7. 10 to the base 3.
8. 10 to the base 5.
9. 75 to the base 3.
10. 13 to the base 20.
11. 130 to the base 20.
12. 1300 to the base 20.

139. Graph of $y = \log_a x$ $(a > 1)$. A general notion of the value of the logarithm of any number can be easily fixed by reference to the graph of $y = \log_a x$. This graph is also the graph of $x = a^y$. In the graph (Fig. 7) we take $a = e = 2.718\ldots$,

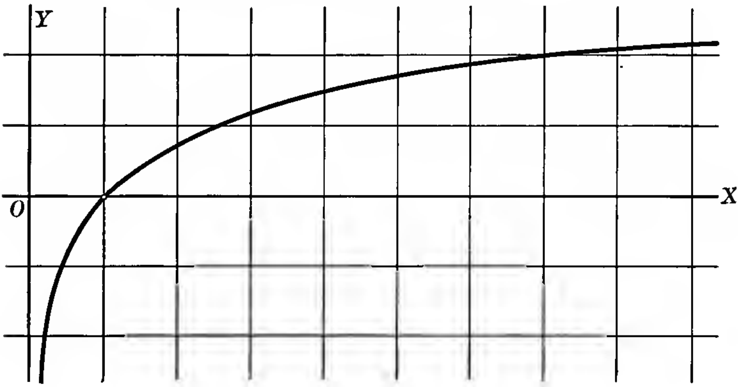

FIG. 7.

but the general form of the curve is not changed if a be given any other positive value greater than 1. If the student retains this picture, he should find it easy to keep in mind the following facts when the base is greater than unity.

1. A negative number does not have a real number for its logarithm.

2. The logarithm of a positive number is positive or negative according as the number is greater than or less than 1.

3. If x approaches zero, $\log x$ decreases without limit.

4. If x increases indefinitely, $\log x$ increases without limit.

EXERCISES

1. Plot the graph of $y = \log_{10} x$ by using tables to find $\log_{10} x$.

2. Plot the graph of $y = \log_5 x$.

HINT:
$$\log_5 x = \frac{\log_{10} x}{\log_{10} 5}.$$

3. Plot the graph of $x = \log_5 y$.

4. Plot the graph of $x = \log_2 y$.

140. Exponential and logarithmic equations. An equation which involves the unknown or unknowns in the exponents is often called an **exponential equation.** Thus, $2^x = 16$ is an exponential equation in x. In this simple example, the value of x can be obtained by inspection; but a table of logarithms is, in general, of value in solving exponential equations.

Such equations arise in a variety of problems. For instance, at compound interest, the amount of one dollar at a nominal rate of 0.05 per annum is (see problem 34, p. 232)

$$A = \left(1 + \frac{0.05}{n}\right)^{nt} \text{ dollars,}$$

in which t is the time in years and n is the number of times per year that interest is converted. We may also write

$$A = \left[\left(1 + \frac{0.05}{n}\right)^{\frac{n}{0.05}}\right]^{0.05t}.$$

When n is increased beyond bound, the interest is said to be converted continuously. It turns out that, in this case (see Arts. 20 and 142),

$$A = e^{0.05t},$$

where e is the base of natural logarithms.

EXAMPLE: What will \$1000 amount to in one year at 5% interest converted continuously?

SOLUTION: Let S be the amount of \$1000 at the end of a year, then

$$S = 1000\, e^{0.05},$$

$$\log S = \log 1000 + 0.05 \log e = 3.0217,$$

$$S = \$1051.$$

An equation which involves the logarithm of an expression that contains an unknown is sometimes called a **logarithmic equation.** Thus,

$$\log_{10} 2x = 3$$

is a logarithmic equation. To solve this equation, we may write, from the definition of a logarithm,

$$2x = 10^3 = 1000.$$

Hence, $$x = 500.$$

EXERCISES AND PROBLEMS

Solve the following equations for x.

1. $5^x = 10.$

SOLUTION:

$$\text{Since } 5^x = 10,$$
$$\log_{10} 5^x = \log_{10} 10 = 1.$$
$$x \log_{10} 5 = 1.$$
$$x = \frac{1}{\log_{10} 5},$$
$$= \frac{1}{.6990} = 1.431.$$

2. $2^{3x}\, 5^{2x-1} = 4^{5x}\, 3^{x+1}.$

SOLUTION:

$$\log_{10} 2^{3x}\, 5^{2x-1} = \log_{10} 4^{5x}\, 3^{x+1},$$
$$3x \log_{10} 2 + (2x - 1) \log_{10} 5 = 5x \log_{10} 4 + (x + 1) \log_{10} 3,$$
$$= 10x \log_{10} 2 + (x + 1) \log_{10} 3.$$

Transposing and collecting terms, we have

$$x\,(2 \log_{10} 5 - 7 \log_{10} 2 - \log_{10} 3) = \log_{10} 3 + \log_{10} 5.$$

$$x = \frac{\log_{10} 3 + \log_{10} 5}{2 \log_{10} 5 - 7 \log_{10} 2 - \log_{10} 3},$$
$$= \frac{0.4771 + 0.6990}{1.3980 - 2.1070 - 0.4771},$$
$$= -0.9916.$$

3. $16 = \log_{10} x^2$.

SOLUTION: $$16 = \log_{10} x^2, \tag{1}$$

From (1), $$x^2 = 10^{16}, \tag{2}$$

$$x = \pm 10^8. \tag{3}$$

4. $11^x = 7$. *Ans.* $x = 0.8115$.

5. $(0.3)^x = 0.8$.

6. $3^{x^2} = 532$. *Ans.* $x = \pm 2.390$.

7. $5^{(x^2-3x)} = \frac{1}{25}$.

8. $21^{x^2-2x} = 9261$. *Ans.* $x = 3$ or -1.

9. In a geometrical progression, $l = ar^{n-1}$, solve for n in terms of a, l, and r.

10. In a geometrical progression, $s = \dfrac{ar^n - a}{r - 1}$, solve for n in terms of a, r, and s.

Solve for x and y the following systems of equations.

11.
$$5^{x+y} = 82, \tag{1}$$
$$3^{x-y} = 4. \tag{2}$$

SOLUTION: From (1) and (2),
$$(x + y) \log 5 = \log 82, \tag{3}$$
$$(x - y) \log 3 = \log 4. \tag{4}$$

Solving the linear equations (3) and (4) for x and y, we get

$$x = \frac{\log 82}{2 \log 5} + \frac{\log 4}{2 \log 3}, \tag{5}$$

$$y = \frac{\log 82}{2 \log 5} - \frac{\log 4}{2 \log 5}. \tag{6}$$

Complete by computing the value of (5) and (6) to three decimal places by the use of tables.

12. $2^{x+y} = 18$,
$3^x = 2^y$.

13. $4^{x+y} = 6^{2x}$,
$\log (x + 1) = \log (y - 3)$.

14. In how many years will \$1000 amount to \$2000 at a nominal rate of 0.06 per annum, (1) when interest is converted annually, (2) when it is converted quarterly, (3) when it is converted continuously? *Ans.* (1) 11.90, (2) 11.58, (3) 11.55. With a seven-place table, the following more accurate results are obtained: (1) 11.89, (2) 11.64, (3) 11.55.

15. Solve for x the equation $e^x + e^{-x} = y$; (*a*) when $y = 2$, (*b*) when $y = 4$.

16. Solve the equation $\log_e (3x + 1) = 2$ for x.

17. Solve the equation $\log_{10} (x^2 - 21x) = 2$ for x. *Ans.* $x = 25$ and -4.

141. The series equal to e. The base of natural logarithms, e, may be defined as

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n, \tag{1}$$

where $\lim\limits_{n\to\infty}$ stands for "the limit when n becomes infinite."

We now proceed to find a series equal to (1), from which we can calculate e to any desired degree of approximation. Since $\frac{1}{n}<1$ when $n>1$, we have, by the binomial theorem, Art. 28,

$$\left(1+\frac{1}{n}\right)^n=1+n\cdot\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\frac{1}{n^3}+\dots$$

$$=1+1+\frac{1-\frac{1}{n}}{1\cdot 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\quad 2\quad 3}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{1\quad 2\cdot 3\cdot 4}+\dots \tag{1}$$

When n becomes infinite, $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, $\frac{4}{n}$, .. each approaches zero as a limit, and we have

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=1+1+\frac{1}{2!}^*+\frac{1}{3!}+\frac{1}{4!}+\dots\text{ ad infinitum.} \tag{2}$$

Putting $\frac{1}{n}=m$ in (2), it follows that

$$\lim_{m\to 0}\left(1+m\right)^{\frac{1}{m}}=e.$$

It follows at once from (2) that $e>2$.

It is easily shown from (2) that

$$e<3;$$

for, since

$$\frac{1}{3!}<\frac{1}{2\cdot 2}$$

* The symbol $r!$, read "factorial r," is used for the product $1\quad 2\cdot 3\ \dots\ r$. Thus, $3!=1\quad 2\cdot 3=6$, $7!=1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7=5040$.

$$\frac{1}{4!} < \frac{1}{2 \cdot 2 \cdot 2}$$

$$\frac{1}{5!} < \frac{1}{2 \cdot 2 \cdot 2 \quad 2}$$

$$\frac{1}{r!} < \frac{1}{2^{r-1}},$$

we have

$$e < 1 + 1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \ldots . \tag{3}$$

The geometrical progression

$$1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \ldots .$$

is the right-hand member of (3) with the first term lacking. The sum of this progression is

$$\frac{1}{1 - \frac{1}{2}} = 2.$$

Hence,

$$e < 1 + 2$$
$$< 3.$$

Exercise: Find the sum of the first 13 terms of the series (2) equal to e and compare with value of e, Art. 138.

142. Exponential series. It is our purpose here to discuss two important series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ ad infinitum,} \tag{1}$$

and

$$a^x = 1 + x\log_e a + \frac{x^2}{2!}\left(\log_e a\right)^2 + \frac{x^3}{3!}\left(\log_e a\right)^3 + \ \ldots \text{ad infinitum.} \tag{2}$$

We use in this discussion the fundamental relation * that

$$e^x = \left[\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n\right]^x = \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^{nx}. \tag{3}$$

* We offer here no proof of this proposition which we merely assume from the theories of limits and exponents.

When $|n| > 1$, we expand $\left(1+\frac{1}{n}\right)^{nx}$ by the binomial theorem. This gives

$$1 + nx \cdot \frac{1}{n} + \frac{nx(nx-1)}{2!}\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{3!}\frac{1}{n^3} + \ldots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{2!} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!} + \ldots. \tag{4}$$

Since $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots$ each approaches 0 as a limit when n becomes infinite, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots. \tag{5}$$

Whatever positive number a may be, we may write

$$a = e^c, \text{ so that } \log_e a = c. \tag{6}$$

Hence, $$a^x = e^{cx} = 1 + cx + \frac{c^2x^2}{2!} + \frac{c^3x^3}{3!} + \frac{c^4x^4}{4!} + \ldots. \tag{7}$$

by substitution of cx for x in (5).

Hence, from (6) and (7), we have

$$a^x = 1 + x\log_e a + \frac{x^2}{2!}\left(\log_e a\right)^2 + \frac{x^3}{3!}\left(\log_e a\right)^3 + \ldots.$$

143. Logarithmic series. The following series is very important for our purposes both because it lies at the basis of the calculation of logarithms and because it gives us valuable approximations in certain complicated interest calculations:

$$\log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots \tag{1}$$

In (2), Art. 142, put $a = 1 + y$, then we have

$$(1+y)^x = 1 + x\log_e(1+y) + \frac{x^2}{2!}\left[\log_e(1+y)\right]^2 + \ldots \tag{2}$$

When y is numerically less than 1, we may expand $(1+y)^x$ by the binomial theorem and obtain

$$(1+y)^x = 1 + x\cdot y + \frac{x(x-1)}{2!}y^2 + \frac{x(x-1)(x-2)}{3!}y^3 + \dots \quad (3)$$

The two series making up the right-hand members of (2) and (3) are equal and convergent * for all values of x. Hence we may equate † coefficients of like powers of x. Thus we have, when we equate the coefficients of the first power of x,

$$\log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \text{ ad inf.} \quad (4)$$

144. Calculation of logarithms to base e. If we put $y = 1$ in the logarithmic series, (1) Art. 143, we have

$$\log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \dots \quad (1)$$

If we put $y = \frac{1}{2}$, we have

$$\log_e \tfrac{3}{2} = \tfrac{1}{2} - \tfrac{1}{2}\cdot\tfrac{1}{2^2} + \tfrac{1}{3}\cdot\tfrac{1}{2^3} - \tfrac{1}{4}\cdot\tfrac{1}{2^4} + \dots$$

or $$\log_e 3 = \log_e 2 + \tfrac{1}{2} - \tfrac{1}{2}\cdot\tfrac{1}{2^2} + \tfrac{1}{3}\cdot\tfrac{1}{2^3} - \tfrac{1}{4}\cdot\tfrac{1}{2^4} + \dots \quad (2)$$

If we put $y = \frac{1}{3}$, we have

$$\log_e \tfrac{4}{3} = \tfrac{1}{3} - \tfrac{1}{2}\cdot\tfrac{1}{3^2} + \tfrac{1}{3}\cdot\tfrac{1}{3^3} - \tfrac{1}{4}\cdot\tfrac{1}{3^4} + \dots$$

or $$\log_e 4 = \log_e 3 + \tfrac{1}{3} - \tfrac{1}{2}\cdot\tfrac{1}{3^2} + \tfrac{1}{3}\cdot\tfrac{1}{3^3} - \tfrac{1}{4}\cdot\tfrac{1}{3^4} + \dots \quad (3)$$

It is clear from this that we could, by taking a sufficient number of terms, calculate $\log_e 2$, $\log_e 3$, $\log_e 4$, and by making $y = \frac{1}{5}, \frac{1}{6}, \dots$, we could similarly find $\log_e 6$, $\log_e 7, \dots$.

It would be found that a large number of terms would have to be taken to obtain the accuracy necessary for many purposes. We may by a transformation of (1) Art. 143, derive a much more convenient series.

Thus, given

$$\log_e(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots, \quad (4)$$

* For meaning and tests of convergence, see Chapter XX, Rietz and Crathorne, *College Algebra*, Edition 1919.

† See Fine, *College Algebra*, Theorem 3, p. 540.

we change the sign of y and obtain

$$\log_e(1 - y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots \tag{5}$$

By subtraction of (5) from (4), we have

$$\log_e(1 + y) - \log_e(1 - y) = 2(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{1}{7}y^7 + \ldots),$$

or $$\log_e \frac{1 + y}{1 - y} = 2(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{1}{7}y^7 + \ldots). \tag{6}$$

Let $$\frac{1 + y}{1 - y} = \frac{z + 1}{z},$$

then $y = \dfrac{1}{2z + 1}$, and (6) becomes

$$\log_e \frac{(z + 1)}{z} = 2\left(\frac{1}{2z + 1} + \frac{1}{3} \cdot \frac{1}{(2z + 1)^3} + \frac{1}{5} \cdot \frac{1}{(2z + 1)^5} + \ldots\right)$$

or

$$\log_e(z+1) = \log_e z + 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \ldots\right). \tag{7}$$

EXERCISES

1. Put $z = 1$ in (7), and obtain $\log_e 2$.

2. Put $z = 2$ in (7), and obtain $\log_e 3$ by using value of $\log_e 2$ from exercise 1. Note that from (7) we obtain the logarithm of any positive integer when we know the logarithm of the next smaller integer.

3. Why is it necessary to calculate from (7) only the logarithms of prime numbers?

4. Given $\log_e 2$ from exercise 1, find $\log_e 17$.

145. Logarithms to base 10. It has been shown in Art. 138 of this chapter that the logarithm of a number N to the base 10 may be obtained by multiplying its logarithm to the base e by the number $M = 0.4342945$ called the modulus. It is thus a very simple matter to convert the logarithm of any number to base e into its logarithm to base 10.

CHAPTER XII

PROGRESSIONS

146. Introduction. In the development of the mathematics of finance, we have found it desirable to use both arithmetical and geometrical progressions. For the benefit of those students who are unfamiliar with progressions, we present in this chapter a very brief treatment of the subject.

147. Arithmetical progressions. An **arithmetical progression** is a sequence of numbers each of which differs from the next preceding one by a fixed number called the **common difference.** Thus,

$$2, 4, 6, 8, \ldots$$

is an arithmetical progression with the common difference 2. In the arithmetical progression

$$10, 8, 6, 4, 2, \ldots$$

the common difference is -2.

The numbers of the sequence are called the **terms** of the progression.

148. Elements of an arithmetical progression. Let a represent the first term, d the common difference, n the number of terms considered, l the nth, or last term, and s the sum of the sequence. The five numbers a, d, n, l, and s are called the **elements** of the arithmetical progression.

149. Relations among the elements. Since a is the first term, we have, by definition of an arithmetical progression,

$$a + d = \text{second term},$$

$$a + 2\,d = \text{third term},$$

$$a + 3d = \text{fourth term},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a + (n - 1)d = n\text{th term}.$$

That is
$$\boldsymbol{l = a + (n - 1)d.} \tag{1}$$

The sum of a finite arithmetical progression may be written in each of the following forms:

$$s = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l,$$

$$s = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a.$$

By addition

$$2s = (a + l) + (a + l) + (a + l) + \ldots + (a + l) + (a + l) + (a + l)$$
$$= n(a + l).$$

Therefore,
$$\boldsymbol{s = \frac{n}{2}(a + l).} \tag{2}$$

Whenever any three of the five elements are given, equations (1) and (2) make it possible to find the remaining two elements.

EXERCISES

Find l and s for the following sequences:

1. 2, 11, 20,...to 10 terms.

SOLUTION: $l = a + (n - 1)\,d.$

Here, $a = 2,\ d = 9,\ n = 10.$

$$l = 2 + 9 \cdot 9 = 83.$$

$$s = \frac{n}{2}(a + l)$$

$$= 5\,(2 + 83) = 425.$$

2. 3, 8, 13, 18, to 10 terms. *Ans.* 48; 255.

3. $-2, -4, -6, -8$, to 12 terms. *Ans.* -24; -156.

4. 66, 65, 64,...to 66 terms. *Ans.* 1; 2211.

150. Geometrical progressions. A **geometrical progression** is a sequence of numbers in which the same quotient is obtained by dividing any term by the preceding term. This quotient is called the **ratio.** Thus,

$$3, 6, 12, 24, \ldots$$

is a geometrical progression with a ratio 2.

151. Elements of a geometrical progression. The elements are the same as those for an arithmetical progression with one exception. Instead of the common difference of an arithmetical progression, we have here a ratio represented by r.

152. Relations among the elements. If a represents the first term, then

$$ar = \text{second term},$$

$$ar^2 = \text{third term},$$

$$ar^3 = \text{fourth term},$$

$$\cdot \quad \cdot \quad \cdot$$

$$ar^{n-1} = n\text{th term}.$$

That is,
$$\boldsymbol{l = ar^{n-1}}. \tag{1}$$

By definition,

$$s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}, \tag{2}$$

Then,
$$sr = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n. \tag{3}$$

Subtracting members of (2) from members of (3), we have

$$sr - s = ar^n - a.$$

Hence,
$$\boldsymbol{s = \frac{ar^n - a}{r - 1} = \frac{a(1 - r^n)}{1 - r}}. \tag{4}$$

Since $l = ar^{n-1}$, (4) may be written in the form

$$s = \frac{rl - a}{r - 1}. \tag{5}$$

Here, as in an arithmetical progression, whenever any three of the five elements are given, relations (1) and (5) make it possible to find the other two.

EXERCISES

1. Given $a = 1$, $r = 3$, $n = 9$; find l and s. *Ans.* 6561; 9841.

2. Given $a = -2$, $r = 3$, $n = 8$; find l and s. *Ans.* -4374; -6560.

3. Given $a = \frac{1}{2}$, $r = \frac{1}{2}$, $n = 10$; find l and s. *Ans.* $\frac{1}{1024}$; $\frac{1023}{1024}$.

4. Given $a = 6$, $r = -2$, $n = 8$; find l and s.

5. Find the sum of

$$v + v^2 + v^3 + v^4 + \ldots \text{to 10 terms,}$$

$$\text{where } v = \frac{1}{1.05}. \qquad \textit{Ans. } 7.72+.$$

153. Number of terms infinite. Consider the geometrical progression

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots.$$

It may at first thought appear that the sum of the first n terms of this progression could be made to exceed any finite number previously assigned by making n large enough. That this is not the case and that the sum can never exceed unity, will be seen from the following illustration. Conceive a particle moving in a straight line towards a point one unit distant in such a way as to describe ½ the distance in the 1st second, ½ the remaining distance in the 2d second, ½ the remaining distance in the 3d second, and so on indefinitely. This is represented in Fig. 8.

The distance AB represents one unit of distance. In the 1st second the particle moves from A to P_1. In the 2d second it moves from P_1 to P_2, and so on. The total distance traversed by the particle in n seconds is given by the sum

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots \text{ to } n \text{ terms,}$$

which sum cannot exceed nor equal 1, no matter how many terms we take, but can be made to differ from 1 by as small a positive number as we please by making the number of terms

large enough. Thus, when $n = 10$, the sum is $\frac{1023}{1024}$ (problem 3, Art. 152). In this illustration, 1 is said to be the limiting * value

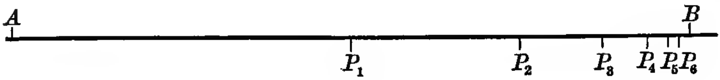

FIG. 8.

of the sum of the first n terms of the series. If s_n represents the sum of the first n terms of the series, we write

$$s = \lim_{n \to \infty} s_n = 1,$$

which reads, "the limit of s_n as n increases **beyond bound** is 1." †

The limit s is called the **sum** of the geometrical progression with infinitely many terms.

For any geometrical progression in which the ratio is less than 1, the above argument can be repeated, and it can be shown that there is a limiting value to the sum of the first n terms of such a series. In Art. 152, we have shown that the sum of the geometrical progression

$$a + ar + ar^2 + \ldots + ar^{n-1}$$

is given by
$$s_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

We may then write

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a}{1 - r} - \lim_{n \to \infty} \frac{ar^n}{1 - r}. \ddagger$$

But
$$\lim_{n \to \infty} \frac{ar^n}{1 - r} \S = 0 \text{ when } |r| < 1.$$

* For definition of limit, see Art. 150, Rietz and Crathorne, *College Algebra* (revised edition).

† The symbol "$n \to \infty$" stands for "n increases beyond bound," or its equivalent, "n becomes infinite."

‡ See Rietz and Crathorne, *College Algebra* (revised edition), Art. 152.

§ Same, Art. 154.

Hence,
$$s = \lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

EXERCISES

Find the sum of the following:

1. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots$ with infinitely many terms. *Ans.* $\frac{2}{3}$.

2. $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots$ with infinitely many terms.. *Ans.* $\frac{4}{3}$.

MISCELLANEOUS PROBLEMS

1. A sum of money S is at simple interest for 78 months, an equal sum for 77 months, an equal sum for 76 months, and so on until the last sum is at interest only one month. The total interest is equal to simple interest for how many months on the sum S? *Ans.* 3081.

2. A man deposits \$100 in a savings bank each six months. He receives interest at the rate of 4 per cent per annum payable semiannually, and deposits this interest in the savings bank immediately upon receiving it. What sum will have accumulated just after he makes his 40th payment?

3. An employer hires a clerk for 10 years at a beginning salary of \$600 per year with either a rise of \$100 each year after the first, or a rise of \$25 each six months after the first half year. Which is the better proposition for the clerk?

4. What distance will an elastic ball traverse before coming to rest if it be dropped from a height of 30 feet and if after each fall it rebounds one-third of the height from which it falls? *Ans.* 60 feet.

TABLES

TABLE I. AMOUNT OF 1

$$s = (1 + i)^n$$

n	½ per cent	1 per cent	1¼ per cent	1½ per cent	1¾ per cent
1	1.005 0000	1.010 0000	1.012 5000	1.015 0000	1.017 5000
2	1.010 0250	1.020 1000	1.025 1562	1.030 2250	1.035 3062
3	1.015 0751	1.030 3010	1.037 9707	1.045 6784	1.053 4241
4	1.020 1505	1.040 6040	1.050 9453	1.061 3636	1.071 8590
5	1.025 2512	1.051 0100	1.064 0822	1.077 2840	1.090 6166
6	1.030 3775	1.061 5202	1.077 3832	1.093 4433	1.109 7024
7	1.035 5294	1.072 1354	1.090 8505	1.109 8449	1.129 1222
8	1.040 7070	1.082 8567	1.104 4861	1.126 4926	1.148 8818
9	1.045 9106	1.093 6853	1.118 2922	1.143 3900	1.168 9872
10	1.051 1401	1.104 6221	1.132 2708	1.160 5408	1.189 4445
11	1.056 3958	1.115 6684	1.146 4242	1.177 9489	1.210 2598
12	1.061 6778	1.126 8250	1.160 7545	1.195 6182	1.231 4393
13	1.066 9862	1.138 0933	1.175 2640	1.213 5524	1.252 9895
14	1.072 3211	1.149 4742	1.189 9548	1.231 7557	1.274 9168
15	1.077 6827	1.160 9690	1.204 8292	1.250 2321	1.297 2279
16	1.083 0712	1.172 5786	1.219 8896	1.268 9856	1.319 9294
17	1.088 4865	1.184 3044	1.235 1382	1.288 0203	1.343 0281
18	1.093 9289	1.196 1475	1.250 5774	1.307 3406	1.366 5311
19	1.099 3986	1.208 1090	1.266 2096	1.326 9508	1.390 4454
20	1.104 8956	1.220 1900	1.282 0372	1.346 8550	1.414 7782
21	1.110 4201	1.232 3919	1.298 0627	1.367 0578	1.439 5368
22	1.115 9722	1.244 7159	1.314 2885	1.387 5637	1.464 7287
23	1.121 5520	1.257 1630	1.330 7171	1.408 3772	1.490 3615
24	1.127 1598	1.269 7346	1.347 3510	1.429 5028	1.516 4428
25	1.132 7956	1.282 4320	1.364 1929	1.450 9454	1.542 9805
26	1.138 4596	1.295 2563	1.381 2454	1.472 7095	1.569 9827
27	1.144 1518	1.308 2089	1.398 5109	1.494 8002	1.597 4574
28	1.149 8726	1.321 2910	1.415 9923	1.517 2222	1.625 4129
29	1.155 6220	1.334 5039	1.433 6922	1.539 9805	1.653 8576
30	1.161 4001	1.347 8489	1.451 6134	1.563 0802	1.682 8001
31	1.167 2071	1.361 3274	1.469 7585	1.586 5264	1.712 2491
32	1.173 0431	1.374 9407	1.488 1305	1.610 3243	1.742 2135
33	1.178 9083	1.388 6901	1.506 7321	1.634 4792	1.772 7022
34	1.184 8029	1.402 5770	1.525 5663	1.658 9964	1.803 7245
35	1.190 7269	1.416 6028	1.544 6359	1.683 8813	1.835 2897
36	1.196 6805	1.430 7688	1.563 9438	1.709 1395	1.867 4073
37	1.202 6639	1.445 0765	1.583 4931	1.734 7766	1.900 0869
38	1.208 6772	1.459 5272	1.603 2868	1.760 7983	1.933 3384
39	1.214 7206	1.474 1225	1.623 3279	1.787 2102	1.967 1718
40	1.220 7942	1.488 8637	1.643 6195	1.814 0184	2.001 5973
41	1.226 8982	1.503 7524	1.664 1647	1.841 2287	2.036 6253
42	1.233 0327	1.518 7899	1.684 9668	1.868 8471	2.072 2662
43	1.239 1979	1.533 9778	1.706 0288	1.896 8798	2.108 5309
44	1.245 3938	1.549 3176	1.727 3542	1.925 3330	2.145 4302
45	1.251 6208	1.564 8108	1.748 9461	1.954 2130	2.182 9752
46	1.257 8789	1.580 4588	1.770 8080	1.983 5262	2.221 1773
47	1.264 1683	1.596 2634	1.792 9431	2.013 2791	2.260 0479
48	1.270 4892	1.612 2261	1.815 3548	2.043 4783	2.299 5987
49	1.276 8416	1.628 3483	1.838 0468	2.074 1305	2.339 8417
50	1.283 2258	1.644 6318	1.861 0224	2.105 2424	2.380 7889
60	1.348 8502	1.816 6967	2.107 1814	2.443 2198	2.831 8163
70	1.417 8305	2.006 7634	2.385 9000	2.835 4563	3.368 2883
80	1.490 3386	2.216 7152	2.701 4849	3.290 6628	4.006 3919
90	1.566 5547	2.448 6327	3.058 8126	3.818 9485	4.765 3808
100	1.646 6685	2.704 8138	3.463 4043	4.432 0456	5.668 1559

TABLE I. AMOUNT OF 1—*Continued*

$s = (1 + i)^n$

n	2 per cent	$2\frac{1}{8}$ per cent	$2\frac{1}{4}$ per cent	$2\frac{3}{8}$ per cent	$2\frac{1}{2}$ per cent
1	1.020 0000	1.021 2500	1.022 5000	1.023 7500	1.025 0000
2	1.040 4000	1.042 9516	1.045 5062	1.048 0641	1.050 6250
3	1.061 2080	1.065 1143	1.069 0301	1.072 9556	1.076 8906
4	1.082 4322	1.087 7480	1.093 0833	1.098 4383	1.103 8129
5	1.104 0808	1.110 8626	1.117 6777	1.124 5262	1.131 4082
6	1.126 1624	1.134 4684	1.142 8254	1.151 2337	1.159 6934
7	1.148 6857	1.158 5759	1.168 5390	1.178 5755	1.188 6858
8	1.171 6594	1.183 1956	1.194 8311	1.206 5666	1.218 4029
9	1.195 0926	1.208 3385	1.221 7148	1.235 2226	1.248 8630
10	1.218 9944	1.234 0157	1.249 2034	1.264 5592	1.280 0845
11	1.243 3743	1.260 2386	1.277 3105	1.294 5924	1.312 0867
12	1.268 2418	1.287 0186	1.306 0500	1.325 3390	1.344 8888
13	1.293 6066	1.314 3678	1.335 4361	1.356 8158	1.378 5110
14	1.319 4788	1.342 2981	1.365 4834	1.389 0402	1.412 9738
15	1.345 8683	1.370 8219	1.396 2068	1.422 0299	1.448 2982
16	1.372 7857	1.399 9519	1.427 6215	1.455 8031	1.484 5056
17	1.400 2414	1.429 7009	1.459 7429	1.490 3784	1.521 6183
18	1.428 2462	1.460 0820	1.492 5872	1.525 7749	1.559 6587
19	1.456 8112	1.491 1088	1.526 1704	1.562 0120	1.598 6502
20	1.485 9474	1.522 7948	1.560 5092	1.599 1098	1.638 6164
21	1.515 6663	1.555 1542	1.595 6207	1.637 0887	1.679 5818
22	1.545 9797	1.588 2012	1.631 5221	1.675 9696	1.721 5714
23	1.576 8993	1.621 9505	1.668 2314	1.715 7738	1.764 6107
24	1.608 4372	1.656 4170	1.705 7666	1.756 5235	1.808 7260
25	1.640 6060	1.691 6158	1.744 1463	1.798 2409	1.853 9441
26	1.673 4181	1.727 5627	1.783 3896	1.840 9491	1.900 2927
27	1.706 8865	1.764 2734	1.823 5159	1.884 6716	1.947 8000
28	1.741 0242	1.801 7642	1.864 5450	1.929 4326	1.996 4950
29	1.775 8447	1.840 0517	1.906 4972	1.975 2566	2.046 4074
30	1.811 3616	1.879 1528	1.949 3934	2.022 1690	2.097 5676
31	1.847 5888	1.919 0848	1.993 2548	2.070 1955	2.150 0068
32	1.884 5406	1.959 8653	2.038 1030	2.119 3626	2.203 7569
33	1.922 2314	2.001 5124	2.083 9603	2.169 6975	2.258 8509
34	1.960 6760	2.044 0446	2.130 8494	2.221 2278	2.315 3221
35	1.999 8896	2.087 4805	2.178 7936	2.273 9820	2.373 2052
36	2.039 8873	2.131 8395	2.227 8164	2.327 9890	2.432 5353
37	2.080 6851	2.177 1411	2.277 9423	2.383 2788	2.493 3487
38	2.122 2988	2.223 4053	2.329 1960	2.439 8816	2.555 6824
39	2.164 7448	2.270 6527	2.381 6029	2.497 8288	2.619 5745
40	2.208 0397	2.318 9041	2.435 1890	2.557 1523	2.685 0638
41	2.252 2005	2.368 1808	2.489 9807	2.617 8846	2.752 1904
42	2.297 2445	2.418 5046	2.546 0053	2.680 0594	2.820 9952
43	2.343 1894	2.469 8978	2.603 2904	2.743 7108	2.891 5201
44	2.390 0531	2.522 3832	2.661 8644	2.808 8740	2.963 8081
45	2.437 8542	2.575 9838	2.721 7564	2.875 5847	3.037 9033
46	2.486 6113	2.630 7235	2.782 9959	2.943 8798	3.113 8509
47	2.536 3435	2.686 6263	2.845 6133	3.013 7970	3.191 6971
48	2.587 0704	2.743 7172	2.909 6396	3.085 3747	3.271 4896
49	2.638 8118	2.802 0211	2.975 1065	3.158 6523	3.353 2768
50	2.691 5880	2.861 5641	3.042 0464	3.233 6703	3.437 1087
60	3.281 0308	3.531 2151	3.800 1348	4.089 1674	4.399 7898
70	3.999 5582	4.357 5750	4.747 1414	5.170 9940	5.632 1029
80	4.875 4392	5.377 3161	5.930 1453	6.539 0278	7.209 5678
90	5.943 1331	6.635 6926	7.407 9578	8.268 9874	9.228 8563
100	7.244 6461	8.188 5490	9.254 0463	10.456 6236	11.813 7164

TABLE I. AMOUNT OF 1—*Continued*

$$s = (1 + i)^n$$

n	3 per cent	$3\frac{1}{2}$ per cent	4 per cent	$4\frac{1}{4}$ per cent	$4\frac{1}{2}$ per cent
1	1.030 0000	1.035 0000	1.040 0000	1.042 5000	1.045 0000
2	1.060 9000	1.071 2250	1.081 6000	1.086 8062	1.092 0250
3	1.092 7270	1.108 7179	1.124 8640	1.132 9955	1.141 1661
4	1.125 5088	1.147 5230	1.169 8586	1.181 1478	1.192 5186
5	1.159 2741	1.187 6863	1.216 6529	1.231 3466	1.246 1819
6	1.194 0523	1.229 2553	1.265 3190	1.283 6788	1.302 2601
7	1.229 8739	1.272 2793	1.315 9318	1.338 2352	1.360 8618
8	1.266 7701	1.316 8090	1.368 5690	1.395 1102	1.422 1006
9	1.304 7732	1.362 8974	1.423 3118	1.454 4024	1.486 0951
10	1.343 9164	1.410 5988	1.480 2443	1.516 2145	1.552 9694
11	1.384 2339	1.459 9697	1.539 4541	1.580 6536	1.622 8530
12	1.425 7609	1.511 0687	1.601 0322	1.647 8314	1.695 8814
13	1.468 5337	1.563 9561	1.665 0735	1.717 8642	1.772 1961
14	1.512 5897	1.618 6945	1.731 6764	1.790 8734	1.851 9449
15	1.557 9674	1.675 3488	1.800 9435	1.866 9855	1.935 2824
16	1.604 7064	1.733 9860	1.872 9812	1.946 3324	2.022 3702
17	1.652 8476	1.794 6756	1.947 9005	2.029 0516	2.113 3768
18	1.702 4331	1.857 4892	2.025 8165	2.115 2862	2.208 4788
19	1.753 5060	1.922 5013	2.106 8492	2.205 1859	2.307 8603
20	1.806 1112	1.989 7889	2.191 1231	2.298 9063	2.411 7140
21	1.860 2946	2.059 4315	2.278 7681	2.396 6098	2.520 2412
22	1.916 1034	2.131 5116	2.369 9188	2.498 4658	2.633 6520
23	1.973 5865	2.206 1145	2.464 7155	2.604 6505	2.752 1664
24	2.032 7941	2.283 3285	2.563 3042	2.715 3482	2.876 0138
25	2.093 7779	2.363 2450	2.665 8363	2.830 7505	3.005 4345
25	2.156 5913	2.445 9586	2.772 4698	2.951 0574	3.140 6790
27	2.221 2890	2.531 5671	2.883 3686	3.076 4773	3.282 0096
28	2.287 9277	2.620 1720	2.998 7033	3.207 2276	3.429 7000
29	2.356 5655	2.711 8780	3.118 6514	3.343 5348	3.584 0365
30	2.427 2625	2.806 7937	3.243 3975	3.485 6350	3.745 3181
31	2.500 0804	2.905 0315	3.373 1334	3.633 7745	3.913 8574
32	2.575 0828	3.006 7076	3.508 0588	3.788 2099	4.089 9810
33	2.652 3352	3.111 9424	3.648 3811	3.949 2088	4.274 0302
34	2.731 9053	3.220 8603	3.794 3163	4.117 0502	4.466 3615
35	2.813 8624	3.333 5904	3.946 0890	4.292 0248	4.667 3478
36	2.898 2783	3.450 2661	4.103 9326	4.474 4359	4.877 3785
37	2.985 2267	3.571 0254	4.268 0899	4.664 5994	5.096 8605
38	3.074 7835	3.696 0113	4.438 8134	4.862 8449	5.326 2192
39	3.167 0270	3.825 3717	4.616 3660	5.069 5158	5.565 8991
40	3.262 0378	3.959 2597	4.801 0206	5.284 9702	5.816 3645
41	3.359 8989	4.097 8338	4.993 0614	5.509 5815	6.078 1009
42	3.460 6959	4.241 2580	5.192 7839	5.743 7387	6.351 6155
43	3.564 5168	4.389 7020	5.400 4953	5.987 8476	6.637 4382
44	3.671 4523	4.543 3416	5.616 5151	6.242 3311	6.936 1229
45	3.781 5958	4.702 3586	5.841 1757	6.507 6302	7.248 2484
46	3.895 0437	4.866 9411	6.074 8227	6.784 2044	7.574 4196
47	4.011 8950	5.037 2840	6.317 8156	7.072 5331	7.915 2685
48	4.132 2519	5.213 5890	6.570 5282	7.373 1158	8.271 4556
49	4.256 2194	5.396 0646	6.833 3494	7.686 4732	8.643 6711
50	4.383 9060	5.584 9269	7.106 6834	8.013 1483	9.032 6363
60	5.891 6031	7.878 0909	10.519 6274	12.149 6514	14.027 4079
70	7.917 8219	11.112 8253	15.571 6184	18.421 4773	21.784 1356
80	10.640 8906	15.675 7375	23.049 7991	27.930 9104	33.830 0964
90	14.300 4671	22.112 1760	34.119 3333	42.349 2505	52.537 1053
100	19.218 6320	31.191 4080	50.504 9482	64.210 5462	81.588 5180

TABLE 1. AMOUNT OF 1—*Continued*

$s = (1 + i)^n$

n	4¾ per cent	5 per cent	6 per cent	7 per cent	8 per cent
1	1.047 5000	1.050 0000	1.060 0000	1.070 0000	1.080 0000
2	1.097 2562	1.102 5000	1.123 6000	1.144 9000	1.166 4000
3	1.149 3759	1.157 6250	1.191 0160	1.225 0430	1.259 7120
4	1.203 9713	1.215 5062	1.262 4770	1.310 7960	1.360 4890
5	1.261 1599	1.276 2816	1.338 2256	1.402 5517	1.469 3281
6	1.321 0650	1.340 0956	1.418 5191	1.500 7304	1.586 8743
7	1.383 8156	1.407 1004	1.503 6303	1.605 7815	1.713 8243
8	1.449 5468	1.477 4554	1.593 8481	1.718 1862	1.850 9302
9	1.518 4003	1.551 3282	1.689 4790	1.838 4592	1.999 0046
10	1.590 5243	1.628 8946	1.790 8477	1.967 1514	2.158 9250
11	1.666 0742	1.710 3394	1.898 2986	2.104 8520	2.331 6390
12	1.745 2128	1.795 8563	2.012 1965	2.252 1916	2.518 1701
13	1.828 1104	1.885 6491	2.132 9283	2.409 8450	2.719 6237
14	1.914 9456	1.979 9316	2.260 9040	2.578 5342	2.937 1936
15	2.005 9055	2.078 9282	2.396 5582	2.759 0315	3.172 1691
16	2.101 1860	2.182 8746	2.540 3517	2.952 1638	3.425 9426
17	2.200 9924	2.292 0183	2.692 7728	3.158 8152	3.700 0180
18	2.305 5395	2.406 6192	2.854 3392	3.379 9323	3.996 0195
19	2.415 0526	2.526 9502	3.025 5995	3.616 5275	4.315 7011
20	2.529 7676	2.653 2977	3.207 1355	3.869 6845	4.660 9571
21	2.649 9316	2.785 9626	3.399 5636	4.140 5624	5.033 8337
22	2.775 8034	2.925 2607	3.603 5374	4.430 4017	5.436 5404
23	2.907 6540	3.071 5238	3.819 7497	4.740 5299	5.871 4636
24	3.045 7676	3.225 0999	4.048 9346	5.072 3670	6.341 1807
25	3.190 4415	3.386 3549	4.291 8707	5.427 4326	6.848 4752
26	3.341 9875	3.555 6727	4.549 3830	5.807 3529	7.396 3532
27	3.500 7319	3.733 4563	4.822 3459	6.213 8676	7.988 0615
28	3.667 0167	3.920 1291	5.111 6867	6.648 8384	8.627 1064
29	3.841 2000	4.116 1356	5.418 3879	7.114 2570	9.317 2749
30	4.023 6570	4.321 9424	5.743 4912	7.612 2550	10.062 6569
31	4.214 7807	4.538 0395	6.088 1006	8.145 1129	10.867 6694
32	4.414 9828	4.764 9415	6.453 3867	8.715 2708	11.737 0830
33	4.624 6944	5.003 1885	6.840 5899	9.325 3398	12.676 0496
34	4.844 3674	5.253 3480	7.251 0253	9.978 1135	13.690 1336
35	5.074 4749	5.516 0154	7.686 0868	10.676 5815	14.785 3443
36	5.315 5124	5.791 8161	8.147 2520	11.423 9422	15.968 1718
37	5.567 9993	6.081 4069	8.636 0871	12.223 6181	17.245 6256
38	5.832 4792	6.385 4773	9.154 2524	13.079 2714	18.625 2756
39	6.109 5220	6.704 7512	9.703 5075	13.994 8204	20.115 2977
40	6.399 7243	7.039 9887	10.285 7179	14.974 4578	21.724 5215
41	6.703 7112	7.391 9882	10.902 8610	16.022 6699	23.462 4832
42	7.022 1375	7.761 5876	11.557 0327	17.144 2568	25.339 4819
43	7.355 6890	8.149 6669	12.250 4546	18.344 3548	27.366 6404
44	7.705 0843	8.557 1503	12.985 4819	19.628 4596	29.555 9717
45	8.071 0758	8.985 0078	13.764 6108	21.002 4518	31.920 4494
46	8.454 4519	9.434 2582	14.590 4875	22.472 6234	34.474 0853
47	8.856 0383	9.905 9711	15.465 9167	24.045 7070	37.232 0122
48	9.276 7001	10.401 2696	16.393 8717	25.728 9065	40.210 5731
49	9.717 3434	10.921 3331	17.377 5040	27.529 9300	43.427 4190
50	10.178 9172	11.467 3998	18.420 1543	29.457 0251	46.901 6125
60	16.189 8154	18.679 1859	32.987 6908	57.946 4268	101.257 0637
70	25.750 2954	30.426 4255	59.075 9302	113.989 3922	218.606 4059
80	40.956 4712	49.561 4411	105.795 9935	224.234 3876	471.954 8343
90	65.142 2639	80.730 3650	189.464 5112	441.102 9799	1018.915 0893
100	103.610 3555	131.501 2578	339.302 0835	867.716 3256	2199.761 2563

TABLE II. PRESENT VALUE OF 1

$$v^n = (1+i)^{-n}$$

n	½ per cent	1 per cent	1¼ per cent	1½ per cent	1¾ per cent
1	0.995 0249	0.990 0990	0.987 6543	0.985 2217	0.982 8010
2	0.990 0745	0.980 2960	0.975 4611	0.970 6618	0.965 8978
3	0.985 1488	0.970 5902	0.963 4183	0.956 3170	0.949 2853
4	0.980 2475	0.960 9803	0.951 5243	0.942 1842	0.932 9585
5	0.975 3707	0.951 4657	0.939 7771	0.928 2603	0.916 9125
6	0.970 5181	0.942 0452	0.928 1749	0.914 5422	0.901 1425
7	0.965 6896	0.932 7180	0.916 7159	0.901 0268	0.885 6438
8	0.960 8852	0.923 4832	0.905 3984	0.887 7111	0.870 4116
9	0.956 1047	0.914 3398	0.894 2207	0.874 5922	0.855 4414
10	0.951 3479	0.905 2870	0.883 1809	0.861 6672	0.840 7286
11	0.946 6149	0.896 3237	0.872 2775	0.848 9332	0.826 2689
12	0.941 9053	0.887 4492	0.861 5086	0.836 3874	0.812 0579
13	0.937 2192	0.878 6626	0.850 8727	0.824 0270	0.798 0913
14	0.932 5565	0.869 9630	0.840 3681	0.811 8493	0.784 3649
15	0.927 9169	0.861 3495	0.829 9932	0.799 8515	0.770 8746
16	0.923 3004	0.852 8213	0.819 7464	0.788 0310	0.757 6163
17	0.918 7068	0.844 3775	0.809 6260	0.776 3853	0.744 5860
18	0.914 1362	0.836 0173	0.799 6306	0.764 9116	0.731 7799
19	0.909 5882	0.827 7399	0.789 7587	0.753 6075	0.719 1940
20	0.905 0629	0.819 5445	0.780 0086	0.742 4704	0.706 8246
21	0.900 5601	0.811 4302	0.770 3788	0.731 4980	0.694 6679
22	0.896 0797	0.803 3962	0.760 8680	0.720 6876	0.682 7203
23	0.891 6216	0.795 4418	0.751 4745	0.710 0371	0.670 9782
24	0.887 1857	0.787 5661	0.742 1971	0.699 5439	0.659 4380
25	0.882 7718	0.779 7684	0.733 0341	0.689 2058	0.648 0963
26	0.878 3799	0.772 0480	0.723 9843	0.679 0205	0.636 9497
27	0.874 0099	0.764 4039	0.715 0463	0.668 9857	0.625 9948
28	0.869 6616	0.756 8356	0.706 2185	0.659 0992	0.615 2283
29	0.865 3349	0.749 3422	0.697 4998	0.649 3589	0.604 6470
30	0.861 0297	0.741 9229	0.688 8887	0.639 7624	0.594 2476
31	0.856 7460	0.734 5772	0.680 3839	0.630 3078	0.584 0272
32	0.852 4836	0.727 3041	0.671 9841	0.620 9929	0.573 9825
33	0.848 2424	0.720 1031	0.663 6880	0.611 8157	0.564 1105
34	0.844 0223	0.712 9733	0.655 4943	0.602 7741	0.554 4084
35	0.839 8231	0.705 9142	0.647 4018	0.593 8661	0.544 8731
36	0.835 6449	0.698 9250	0.639 4092	0.585 0897	0.535 5018
37	0.831 4875	0.692 0049	0.631 5152	0.576 4431	0.526 2917
38	0.827 3507	0.685 1534	0.623 7187	0.567 9242	0.517 2400
39	0.823 2346	0.678 3697	0.616 0185	0.559 5313	0.508 3440
40	0.819 1389	0.671 6531	0.608 4133	0.551 2623	0.499 6010
41	0.815 0635	0.665 0031	0.600 9021	0.543 1156	0.491 0083
42	0.811 0085	0.658 4189	0.593 4835	0.535 0892	0.482 5635
43	0.806 9736	0.651 8999	0.586 1566	0.527 1815	0.474 2639
44	0.802 9588	0.645 4455	0.578 9201	0.519 3907	0.466 1070
45	0.798 9640	0.639 0549	0.571 7729	0.511 7149	0.458 0904
46	0.794 9891	0.632 7276	0.564 7140	0.504 1526	0.450 2117
47	0.791 0339	0.626 4630	0.557 7422	0.496 7021	0.442 4685
48	0.787 0984	0.620 2604	0.550 8565	0.489 3617	0.434 8585
49	0.783 1825	0.614 1192	0.544 0558	0.482 1298	0.427 3793
50	0.779 2861	0.608 0388	0.537 3390	0.475 0047	0.420 0288
60	0.741 3722	0.550 4496	0.474 5676	0.409 2960	0.353 1302
70	0.705 3029	0.498 3149	0.419 1290	0.352 6769	0.296 8867
80	0.670 9885	0.451 1179	0.370 1668	0.303 8902	0.249 6011
90	0.638 3435	0.408 3912	0.326 9242	0.261 8522	0.209 8468
100	0.607 2868	0.369 7112	0.288 7333	0.225 6294	0.176 4242

TABLE II. PRESENT VALUE OF 1—*Continued*

$$v^n = (1 + i)^{-n}$$

n	2 per cent	$2^1/_8$ per cent	$2\frac{1}{4}$ per cent	$2^3/_8$ per cent	$2\frac{1}{2}$ per cent
1	0.980 3922	0.979 1922	0.977 9951	0.976 8010	0.975 6098
2	0.961 1688	0.958 8173	0.956 4744	0.954 1402	0.951 8144
3	0.942 3223	0.938 8664	0.935 4273	0.932 0050	0.928 5994
4	0.923 8454	0.919 3306	0.914 8434	0.910 3834	0.905 9506
5	0.905 7308	0.900 2013	0.894 7123	0.889 2634	0.883 8543
6	0.887 9714	0.881 4701	0.875 0243	0.868 6334	0.862 2969
7	0.870 5602	0.863 1286	0.855 7695	0.848 4819	0.841 2652
8	0.853 4904	0.845 1688	0.836 9384	0.828 7980	0.820 7466
9	0.836 7553	0.827 5826	0.818 5216	0.809 5707	0.800 7284
10	0.820 3483	0.810 3624	0.800 5101	0.790 7894	0.781 1984
11	0.804 2630	0.793 5006	0.782 8950	0.772 4439	0.762 1448
12	0.788 4932	0.776 9895	0.765 6675	0.754 5239	0.743 5559
13	0.773 0325	0.760 8221	0.748 8190	0.737 0197	0.725 4204
14	0.757 8750	0.744 9910	0.732 3414	0.719 9216	0.707 7272
15	0.743 0147	0.729 4894	0.716 2263	0.703 2201	0.690 4656
16	0.728 4458	0.714 3103	0.700 4658	0.686 9061	0.673 6249
17	0.714 1626	0.699 4470	0.685 0521	0.670 9705	0.657 1951
18	0.700 1594	0.684 8930	0.669 9776	0.655 4047	0.641 1659
19	0.686 4308	0.670 6419	0.655 2348	0.640 1999	0.625 5277
20	0.672 9713	0.656 6873	0.640 8165	0.625 3479	0.610 2709
21	0.659 7758	0.643 0230	0.626 7154	0.610 8404	0.595 3863
22	0.646 8390	0.629 6431	0.612 9246	0.596 6696	0.580 8647
23	0.634 1559	0.616 5416	0.599 4372	0.582 8274	0.566 6972
24	0.621 7215	0.603 7127	0.586 2467	0.569 3064	0.552 8754
25	0.609 5309	0.591 1508	0.573 3464	0.556 0990	0.539 3906
26	0.597 5793	0.578 8502	0.560 7300	0.543 1981	0.526 2347
27	0.585 8620	0.566 8056	0.548 3912	0.530 5964	0.513 3997
28	0.574 3746	0.555 0116	0.536 3239	0.518 2871	0.500 8778
29	0.563 1123	0.543 4630	0.524 5221	0.506 2633	0.488 6612
30	0.552 0709	0.532 1547	0.512 9801	0.494 5185	0.476 7427
31	0.541 2460	0.521 0817	0.501 6920	0.483 0462	0.465 1148
32	0.530 6333	0.510 2392	0.490 6523	0.471 8400	0.453 7706
33	0.520 2287	0.499 6222	0.479 8556	0.460 8937	0.442 7030
34	0.510 0282	0.489 2261	0.469 2964	0.450 2015	0.431 9053
35	0.500 0276	0.479 0464	0.458 9696	0.439 7572	0.421 3711
36	0.490 2232	0.469 0785	0.448 8700	0.429 5553	0.411 0937
37	0.480 6109	0.459 3180	0.438 9927	0.419 5900	0.401 0670
38	0.471 1872	0.449 7606	0.429 3327	0.409 8559	0.391 2849
39	0.461 9482	0.440 4020	0.419 8853	0.400 3477	0.381 7414
40	0.452 8904	0.431 2382	0.410 6458	0.391 0600	0.372 4306
41	0.444 0102	0.422 2651	0.401 6095	0.381 9878	0.363 3470
42	0.435 3041	0.413 4786	0.392 7722	0.373 1261	0.354 4848
43	0.426 7688	0.404 8750	0.384 1292	0.364 4699	0.345 8389
44	0.418 4007	0.396 4505	0.375 6765	0.356 0146	0.337 4038
45	0.410 1968	0.388 2012	0.367 4098	0.347 7554	0.329 1744
46	0.402 1537	0.380 1236	0.359 3250	0.339 6878	0.321 1458
47	0.394 2684	0.372 2140	0.351 4181	0.331 8074	0.313 3129
48	0.386 5376	0.364 4691	0.343 6852	0.324 1098	0.305 6712
49	0.378 9584	0.356 8852	0.336 1224	0.316 5907	0.298 2158
50	0.371 5279	0.349 4592	0.328 7261	0.309 2461	0.290 9422
60	0.304 7823	0.283 1886	0.263 1486	0.244 5486	0.227 2836
70	0.250 0276	0.229 4854	0.210 6531	0.193 3864	0.177 5536
80	0.205 1097	0.185 9664	0.168 6299	0.152 9279	0.138 7046
90	0.168 2614	0.150 7002	0.134 9900	0.120 9338	0.108 3558
100	0.138 0330	0.122 1218	0.108 0608	0.095 6332	0.084 6474

TABLE II. PRESENT VALUE OF 1—*Continued*

$$v^n = (1+i)^{-n}$$

n	3 per cent	3½ per cent	4 per cent	4¼ per cent	4 ½ per cent
1	0.970 8738	0.966 1836	0.961 5385	0.959 2326	0.956 9378
2	0.942 5959	0.933 5107	0.924 5562	0.920 1272	0.915 7300
3	0.915 1417	0.901 9427	0.888 9964	0.882 6160	0.876 2966
4	0.888 4870	0.871 4422	0.854 8042	0.846 6341	0.838 5613
5	0.862 6088	0.841 9732	0.821 9271	0.812 1190	0.802 4510
6	0.837 4843	0.813 5006	0.790 3145	0.779 0110	0.767 8957
7	0.813 0915	0.785 9910	0.759 9178	0.747 2528	0.734 8285
8	0.789 4092	0.759 4116	0.730 6902	0.716 7893	0.703 1851
9	0.766 4167	0.733 7310	0.702 5867	0.687 5676	0.672 9044
10	0.744 0939	0.708 9188	0.675 5642	0.659 5373	0.643 9277
11	0.722 4213	0.684 9457	0.649 5809	0.632 6497	0.616 1987
12	0.701 3799	0.661 7833	0.624 5970	0.606 8582	0.589 6639
13	0.680 9513	0.639 4042	0.600 5741	0.582 1182	0.564 2716
14	0.661 1178	0.617 7818	0.577 4751	0.558 3868	0.539 9729
15	0.641 8620	0.596 8906	0.555 2645	0.535 6228	0.516 7204
16	0.623 1669	0.576 7059	0.533 9082	0.513 7868	0.494 4693
17	0.605 0164	0.557 2038	0.513 3732	0.492 8411	0.473 1764
18	0.587 3946	0.538 3611	0.493 6281	0.472 7493	0.452 8004
19	0.570 2860	0.520 1557	0.474 6424	0.453 4765	0.433 3018
20	0.553 6758	0.502 5659	0.456 3870	0.434 9894	0.414 6429
21	0.537 5493	0.485 5709	0.438 8336	0.417 2561	0.396 7874
22	0.521 8925	0.469 1506	0.421 9554	0.400 2456	0.379 7009
23	0.506 6918	0.453 2856	0.405 7263	0.383 9287	0.363 3501
24	0.491 9337	0.437 9571	0.390 1215	0.368 2769	0.347 7035
25	0.477 6056	0.423 1470	0.375 1168	0.353 2632	0.332 7306
26	0.463 6947	0.408 8377	0.360 6892	0.338 8616	0.318 4025
27	0.450 1891	0.395 0122	0.346 8166	0.325 0471	0.304 6914
28	0.437 0768	0.381 6543	0.333 4775	0.311 7958	0.291 5707
29	0.424 3464	0.368 7482	0.320 6514	0.299 0847	0.279 0150
30	0.411 9868	0.356 2784	0.308 3187	0.286 8918	0.267 0000
31	0.399 9872	0.344 2304	0.296 4603	0.275 1959	0.255 5024
32	0.388 3370	0.332 5897	0.285 0579	0.263 9769	0.244 4999
33	0.377 0262	0.321 3427	0.274 0942	0.253 2153	0.233 9712
34	0.366 0449	0.310 4760	0.263 5521	0.242 8924	0.223 8959
35	0.355 3834	0.299 9769	0.253 4155	0.232 9903	0.214 2544
36	0.345 0324	0.289 8327	0.243 6687	0.223 4919	0.205 0282
37	0.334 9829	0.280 0316	0.234 2968	0.214 3807	0.196 1992
38	0.325 2262	0.270 5619	0.225 2854	0.205 6409	0.187 7504
39	0.315 7536	0.261 4125	0.216 6206	0.197 2575	0.179 6655
40	0.306 5568	0.252 5725	0.208 2890	0.189 2158	0.171 9287
41	0.297 6280	0.244 0314	0.200 2779	0.181 5020	0.164 5251
42	0.288 9592	0.235 7791	0.192 5749	0.174 1026	0.157 4403
43	0.280 5429	0.227 8059	0.185 1682	0.167 0049	0.150 6605
44	0.272 3718	0.220 1023	0.178 0464	0.160 1966	0.144 1728
45	0.264 4386	0.212 6592	0.171 1984	0.153 6658	0.137 9644
46	0.256 7365	0.205 4679	0.164 6139	0.147 4012	0.132 0233
47	0.249 2588	0.198 5197	0.158 2826	0.141 3921	0.126 3381
48	0.241 9988	0.191 8064	0.152 1948	0.135 6279	0.120 8977
49	0.234 9503	0.185 3202	0.146 3411	0.130 0987	0.115 6916
50	0.228 1071	0.179 0534	0.140 7126	0.124 7949	0.110 7096
60	0.169 7331	0.126 9343	0.095 0604	0.082 3069	0.071 2890
70	0.126 2974	0.089 9861	0.064 2194	0.054 2845	0.045 9050
80	0.093 9771	0.063 7928	0.043 3843	0.035 8026	0.029 5595
90	0.069 9278	0.045 2240	0.029 3089	0.023 6132	0.019 0342
100	0.052 0328	0.032 0601	0.019 8000	0.015 5738	0.012 2566

TABLE II. PRESENT VALUE OF 1—*Continued*

$$v^n = (1 + i)^{-n}$$

n	4¾ per cent	5 per cent	6 per cent	7 per cent	8 per cent
1	0.954 6539	0.952 3810	0.943 3962	0.934 5794	0.925 9259
2	0.911 3641	0.907 0295	0.889 9964	0.873 4387	0.857 3388
3	0.870 0374	0.863 8376	0.839 6193	0.816 2979	0.793 8322
4	0.830 5846	0.822 7025	0.792 0937	0.762 8952	0.735 0298
5	0.792 9209	0.783 5262	0.747 2582	0.712 9862	0.680 5832
6	0.756 9650	0.746 2154	0.704 9605	0.666 3422	0.630 1696
7	0.722 6396	0.710 6813	0.665 0571	0.622 7497	0.583 4904
8	0.689 8708	0.676 8394	0.627 4124	0.582 0091	0.540 2689
9	0.658 5878	0.644 6089	0.591 8985	0.543 9337	0.500 2490
10	0.628 7235	0.613 9132	0.558 3948	0.508 3493	0.463 1935
11	0.600 2134	0.584 6793	0.526 7875	0.475 0928	0.428 8829
12	0.572 9960	0.556 8374	0.496 9694	0.444 0120	0.397 1138
13	0.547 0129	0.530 3214	0.468 8390	0.414 9644	0.367 6979
14	0.522 2080	0.505 0680	0.442 3010	0.387 8172	0.340 4610
15	0.498 5280	0.481 0171	0.417 2651	0.362 4460	0.315 2417
16	0.475 9217	0.458 1115	0.393 6463	0.338 7346	0.291 8905
17	0.454 3405	0.436 2967	0.371 3644	0.316 5744	0.270 2690
18	0.433 7380	0.415 5206	0.350 3438	0.295 8639	0.250 2490
19	0.414 0696	0.395 7340	0.330 5130	0.276 5083	0.231 7121
20	0.395 2932	0.376 8895	0.311 8047	0.258 4190	0.214 5482
21	0.377 3682	0.358 9424	0.294 1554	0.241 5131	0.198 6558
22	0.360 2561	0.341 8499	0.277 5051	0.225 7132	0.183 9405
23	0.343 9199	0.325 5713	0.261 7973	0.210 9469	0.170 3153
24	0.328 3245	0.310 0679	0.246 9786	0.197 1466	0.157 6993
25	0.313 4362	0.295 3028	0.232 9986	0.184 2492	0.146 0179
26	0.299 2231	0.281 2407	0.219 8100	0.172 1955	0.135 2018
27	0.285 6546	0.267 8483	0.207 3680	0.160 9304	0.125 1868
28	0.272 7012	0.255 0936	0.195 6301	0.150 4022	0.115 9137
29	0.260 3353	0.242 9463	0.184 5567	0.140 5628	0.107 3275
30	0.248 5301	0.231 3774	0.174 1101	0.131 3671	0.099 3773
31	0.237 2603	0.220 3595	0.164 2548	0.122 7730	0.092 0160
32	0.226 5014	0.209 8662	0.154 9574	0.114 7411	0.085 2000
33	0.216 2305	0.199 8725	0.146 1862	0.107 2347	0.078 8889
34	0.206 4253	0.190 3548	0.137 9115	0.100 2193	0.073 0453
35	0.197 0647	0.181 2903	0.130 1052	0.093 6629	0.067 6345
36	0.188 1286	0.172 6574	0.122 7408	0.087 5355	0.062 6246
37	0.179 5977	0.164 4356	0.115 7932	0.081 8088	0.057 9857
38	0.171 4537	0.156 6054	0.109 2388	0.076 4569	0.053 6905
39	0.163 6789	0.149 1480	0.103 0555	0.071 4550	0.049 7134
40	0.156 2567	0.142 0457	0.097 2222	0.066 7804	0.046 0309
41	0.149 1711	0.135 2816	0.091 7190	0.062 4116	0.042 6212
42	0.142 4068	0.128 8396	0.086 5274	0.058 3286	0.039 4641
43	0.135 9492	0.122 7044	0.081 6296	0.054 5127	0.036 5408
44	0.129 7844	0.116 8613	0.077 0091	0.050 9464	0.033 8341
45	0.123 8992	0.111 2965	0.072 6501	0.047 6135	0.031 3279
46	0.118 2809	0.105 9967	0.068 5378	0.044 4986	0.029 0073
47	0.112 9173	0.100 9492	0.064 6583	0.041 5875	0.026 8586
48	0.107 7970	0.096 1421	0.060 9984	0.038 8668	0.024 8691
49	0.102 9088	0.091 5639	0.057 5457	0.036 3241	0.023 0269
50	0.098 2423	0.087 2037	0.054 2884	0.033 9478	0.021 3212
60	0.061 7672	0.053 5355	0.030 3143	0.017 2573	0.009 8758
70	0.038 8345	0.032 8662	0.016 9274	0.008 7728	0.004 5744
80	0.024 4162	0.020 1770	0.009 4522	0.004 4596	0.002 1188
90	0.015 3510	0.012 3869	0.005 2780	0.002 2670	0.000 9814
100	0.009 6515	0.007 6045	0.002 9472	0.001 1524	0.000 4546

TABLE III. AMOUNT OF 1 PER ANNUM

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	½ per cent	1 per cent	1¼ per cent	1½ per cent	1¾ per cent
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000
2	2.005 0000	2.010 0000	2.012 5000	2.015 0000	2.017 5000
3	3.015 0250	3.030 1000	3.037 6562	3.045 2250	3.052 8062
4	4.030 1001	4.060 4010	4.075 6270	4.090 9034	4.106 2304
5	5.050 2506	5.101 0050	5.126 5723	5.152 2669	5.178 0894
6	6.075 5019	6.152 0151	6.190 6544	6.229 5509	6.268 7060
7	7.105 8794	7.213 5352	7.268 0376	7.322 9942	7.378 4083
8	8.141 4088	8.285 6706	8.358 8881	8.432 8391	8.507 5304
9	9.182 1158	9.368 5273	9.463 3742	9.559 3317	9.656 4122
10	10.228 0264	10.462 2125	10.581 6664	10.702 7217	10.825 3994
11	11.279 1665	11.566 8347	11.713 9372	11.863 2625	12.014 8439
12	12.335 5624	12.682 5030	12.860 3614	13.041 2114	13.225 1037
13	13.397 2402	13.809 3280	14.021 1159	14.236 8296	14.456 5430
14	14.464 2264	14.947 4213	15.196 3799	15.450 3820	15.709 5325
15	15.536 5475	16.096 8955	16.386 3346	16.682 1378	16.984 4494
16	16.614 2303	17.257 8645	17.591 1638	17.932 3698	18.281 6772
17	17.697 3014	18.430 4431	18.811 0534	19.201 3554	19.601 6066
18	18.785 7879	19.614 7476	20.046 1915	20.489 3757	20.944 6347
19	19.879 7168	20.810 8950	21.296 7689	21.796 7164	22.311 1658
20	20.979 1154	22.019 0040	22.562 9785	23.123 6671	23.701 6112
21	22.084 0110	23.239 1940	23.845 0158	24.470 5221	25.116 3894
22	23.194 4311	24.471 5860	25.143 0785	25.837 5799	26.555 9262
23	24.310 4032	25.716 3018	26.457 3670	27.225 1436	28.020 6549
24	25.431 9552	26.973 4648	27.788 0840	28.633 5208	29.511 0164
25	26.559 1150	28.243 1995	29.135 4351	30.063 0236	31.027 4592
26	27.691 9106	29.525 6315	30.499 6280	31.513 9690	32.570 4397
27	28.830 3702	30.820 8878	31.880 8734	32.986 6785	34.140 4224
28	29.974 5220	32.129 0967	33.279 3843	34.481 4787	35.737 8798
29	31.124 3946	33.450 3877	34.695 3766	35.998 7008	37.363 2927
30	32.280 0166	34.784 8915	36.129 0688	37.538 6814	39.017 1503
31	33.441 4167	36.132 7404	37.580 6822	39.101 7616	40.699 9504
32	34.608 6238	37.494 0678	39.050 4407	40.688 2880	42.412 1996
33	35.781 6669	38.869 0085	40.538 5712	42.298 6123	44.154 4130
34	36.960 5752	40.257 6986	42.045 3033	43.933 0915	45.927 1153
35	38.145 3781	41.660 2756	43.570 8696	45.592 0879	47.730 8398
36	39.336 1050	43.076 8784	45.115 5055	47.275 9692	49.566 1295
37	40.532 7855	44.507 6471	46.679 4493	48.985 1087	51.433 5368
38	41.735 4494	45.952 7236	48.262 9424	50.719 8854	53.333 6236
39	42.944 1267	47.412 2508	49.866 2292	52.480 6837	55.266 9621
40	44.158 8473	48.886 3734	51.489 5571	54.267 8939	57.234 1339
41	45.379 6415	50.375 2371	53.133 1765	56.081 9123	59.235 7312
42	46.606 5397	51.878 9895	54.797 3412	57.923 1410	61.272 3565
43	47.839 5724	53.397 7794	56.482 3080	59.791 9881	63.344 6228
44	49.078 7703	54.931 7572	58.188 3369	61.688 8679	65.453 1537
45	50.324 1642	56.481 0747	59.915 6911	63.614 2010	67.598 5839
46	51.575 7850	58.045 8855	61.664 6372	65.568 4140	69.781 5591
47	52.833 6639	59.626 3443	63.435 4452	67.551 9402	72.002 7364
48	54.097 8322	61.222 6078	65.228 3882	69.565 2193	74.262 7842
49	55.368 3214	62.834 8338	67.043 7431	71.608 6976	76.562 3830
50	56.645 1630	64.463 1822	68.881 7899	73.682 8280	78.902 2247
60	69.770 0305	81.669 6699	88.574 5078	96.214 6517	104.675 2159
70	83.566 1055	100.676 3368	110.871 9978	122.363 7530	135.330 7583
80	98.067 7136	121.671 5217	136.118 7953	152.710 8525	171.793 8242
90	113.310 9358	144.863 2675	164.705 0076	187.929 9004	215.164 6172
100	129.333 6984	170.481 3829	197.072 3420	228.803 0433	266.751 7679

TABLE III. AMOUNT OF 1 PER ANNUM—*Continued*

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	2 per cent	$2^1/_8$ per cent	$2\frac{1}{4}$ per cent	$2^3/_8$ per cent	$2\frac{1}{2}$ per cent
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000
2	2.020 0000	2.021 2500	2.022 5000	2.023 7500	2.025 0000
3	3.060 4000	3.064 2016	3.068 0062	3.071 8141	3.075 6250
4	4.121 6080	4.129 3158	4.137 0364	4.144 7696	4.152 5156
5	5.204 0402	5.217 0638	5.230 1197	5.243 2079	5.256 3285
6	6.308 1210	6.327 9264	6.347 7974	6.367 7341	6.387 7367
7	7.434 2834	7.462 3948	7.490 6228	7.518 9678	7.547 4302
8	8.582 9690	8.620 9707	8.659 1619	8.697 5433	8.736 1159
9	9.754 6284	9.804 1664	9.853 9930	9.904 1099	9.954 5188
10	10.949 7210	11.012 5049	11.075 7078	11.139 3326	11.203 3818
11	12.168 7154	12.246 5206	12.324 9113	12.403 8917	12.483 4663
12	13.412 0897	13.506 7592	13.602 2218	13.698 4841	13.795 5530
13	14.680 3315	14.793 7778	14.908 2718	15.023 8231	15.140 4418
14	15.973 9382	16.108 1456	16.243 7079	16.380 6389	16.518 9528
15	17.293 4169	17.450 4437	17.609 1913	17.769 6791	17.931 9267
16	18.639 2852	18.821 2656	19.005 3981	19.191 7090	19.380 2248
17	20.012 0710	20.221 2175	20.433 0196	20.647 5121	20.864 7304
18	21.412 3124	21.650 9184	21.892 7625	22.137 8905	22.386 3487
19	22.840 5586	23.111 0004	23.385 3497	23.663 6654	23.946 0074
20	24.297 3698	24.602 1092	24.911 5200	25.225 6774	25.544 6576
21	25.783 3172	26.124 9040	26.472 0292	26.824 7873	27.183 2740
22	27.298 9835	27.680 0582	28.067 6499	28.461 8760	28.862 8559
23	28.844 9632	29.268 2594	29.699 1720	30.137 8455	30.584 4273
24	30.421 8625	30.890 2100	31.367 4034	31.853 6194	32.349 0380
25	32.030 2997	32.546 6269	33.073 1700	33.610 1428	34.157 7639
26	33.670 9057	34.238 2427	34.817 3163	35.408 3837	36.011 7080
27	35.344 3238	35.965 8054	36.600 7059	37.249 3328	37.912 0007
28	37.051 2103	37.730 0788	38.424 2218	39.134 0045	39.859 8008
29	38.792 2345	39.531 8429	40.288 7668	41.063 4371	41.856 2958
30	40.568 0792	41.371 8946	42.195 2640	43.038 6937	43.902 7032
31	42.379 4408	43.251 0474	44.144 6575	45.060 8627	46.000 2707
32	44.227 0296	45.170 1321	46.137 9123	47.131 0582	48.150 2775
33	46.111 5702	47.129 9974	48.176 0153	49.250 4208	50.354 0344
34	48.033 8016	49.131 5099	50.259 9756	51.420 1183	52.612 8853
35	49.994 4776	51.175 5544	52.390 8251	53.641 3461	54.928 2074
36	51.994 3672	53.263 0350	54.569 6186	55.915 3281	57.301 4126
37	54.034 2545	55.394 8745	56.797 4351	58.243 3171	59.733 9479
38	56.114 9396	57.572 0156	59.075 3774	60.626 5959	62.227 2966
39	58.237 2384	59.795 4209	61.404 5733	63.066 4776	64.782 9791
40	60.401 9832	62.066 0736	63.786 1762	65.564 3064	67.402 5535
41	62.610 0228	64.384 9776	66.221 3652	68.121 4587	70.087 6174
42	64.862 2233	66.753 1584	68.711 3459	70.739 3433	72.839 8078
43	67.159 4678	69.171 6630	71.257 3512	73.419 4027	75.660 8030
44	69.502 6571	71.641 5609	73.860 6416	76.163 1135	78.552 3231
45	71.892 7103	74.163 9440	76.522 5060	78.971 9875	81.516 1312
46	74.330 5645	76.739 9278	79.244 2624	81.847 5722	84.554 0344
47	76.817 1758	79.370 6513	82.027 2583	84.791 4520	87.667 8853
48	79.353 5193	82.057 2777	84.872 8716	87.805 2490	90.859 5824
49	81.940 5897	84.800 9948	87.782 5113	90.890 6237	94.131 0720
50	84.579 4014	87.603 0160	90.757 6178	94.049 2760	97.484 3488
60	114.051 5394	119.116 0045	124.450 4349	130.070 2048	135.991 5900
70	149.977 9111	158.003 5281	166.539 6176	175.620 8000	185.284 1142
80	193.771 9578	205.991 3438	219.117 5688	233.222 2217	248.382 7126
90	247.156 6563	265.209 0632	284.798 1256	306.062 6266	329.154 2533
100	312.232 3059	338.284 6605	366.846 5021	398.173 6268	432.548 6540

TABLE III. AMOUNT OF 1 PER ANNUM

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	3 per cent	3½ per cent	4 per cent	4¼ per cent	4½ per cent
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000
2	2.030 0000	2.035 0000	2.040 0000	2.042 5000	2.045 0000
3	3.090 9000	3.106 2250	3.121 6000	3.129 3062	3.137 0250
4	4.183 6270	4.214 9429	4.246 4640	4.262 3018	4.278 1911
5	5.309 1358	5.362 4659	5.416 3226	5.443 4496	5.470 7097
6	6.468 4099	6.550 1522	6.632 9755	6.674 7962	6.716 8917
7	7.662 4622	7.779 4075	7.898 2945	7.958 4750	8.019 1518
8	8.892 3360	9.051 6868	9.214 2263	9.296 7102	9.380 0136
9	10.159 1061	10.368 4958	10.582 7953	10.691 8204	10.802 1142
10	11.463 8793	11.731 3932	12.006 1071	12.146 2228	12.288 2094
11	12.807 7957	13.141 9919	13.486 3514	13.662 4372	13.841 1788
12	14.192 0296	14.601 9616	15.025 8055	15.243 0908	15.464 0318
13	15.617 7904	16.113 0303	16.626 8377	16.890 9222	17.159 9133
14	17.086 3242	17.676 9864	18.291 9112	18.608 7864	18.932 1094
15	18.598 9139	19.295 6809	20.023 5876	20.399 6598	20.784 0543
16	20.156 8813	20.971 0297	21.824 5311	22.266 6453	22.719 3367
17	21.761 5877	22.705 0158	23.697 5124	24.212 9778	24.741 7069
18	23.414 4354	24.499 6913	25.645 4129	26.242 0293	26.855 0837
19	25.116 8684	26.357 1805	27.671 2294	28.357 3156	29.063 5625
20	26.870 3745	28.279 6818	29.778 0786	30.562 5015	31.371 4228
21	28.676 4857	30.269 4707	31.969 2017	32.861 4078	33.783 1368
22	30.536 7803	32.328 9022	34.247 9698	35.258 0176	36.303 3780
23	32.452 8837	34.460 4137	36.617 8886	37.756 4834	38.937 0300
24	34.426 4702	36.666 5282	39.082 6041	40.361 1339	41.689 1963
25	36.459 2643	38.949 8567	41.645 9083	43.076 4821	44.565 2102
26	38.553 0422	41.313 1017	44.311 7446	45.907 2326	47.570 6446
27	40.709 6335	43.759 0602	47.084 2144	48.858 2900	50.711 3236
28	42.930 9225	46.290 6273	49.967 5830	51.934 7673	53.993 3332
29	45.218 8502	48.910 7993	52.966 2863	55.141 9949	57.423 0332
30	47.575 4157	51.622 6773	56.084 9378	58.485 5297	61.007 0697
31	50.002 6782	54.429 4710	59.328 3353	61.971 1647	64.752 3878
32	52.502 7585	57.334 5025	62.701 4687	65.604 9392	68.666 2452
33	55.077 8413	60.341 2100	66.209 5274	69.393 1491	72.756 2263
34	57.730 1765	63.453 1524	69.857 9085	73.342 3580	77.030 2565
35	60.462 0818	66.674 0127	73.652 2249	77.459 4082	81.496 6180
36	63.275 9443	70.007 6032	77.598 3138	81.751 4330	86.163 9658
37	66.174 2226	73.457 8693	81.702 2464	86.225 8690	91.041 3443
38	69.159 4493	77.028 8947	85.970 3363	90.890 4684	96.138 2048
39	72.234 2328	80.724 9060	90.409 1497	95.753 3133	101.464 4240
40	75.401 2597	84.550 2778	95.025 5157	100.822 8291	107.030 3231
41	78.663 2975	88.509 5375	99.826 5363	106.107 7993	112.846 6876
42	82.023 1964	92.607 3713	104.819 5978	111.617 3808	118.924 7885
43	85.483 8923	96.848 6293	110.012 3817	117.361 1195	125.276 4040
44	89.048 4091	101.238 3313	115.412 8770	123.348 9671	131.913 8422
45	92.719 8614	105.781 6729	121.029 3920	129.591 2982	138.849 9651
46	96.501 4572	110.484 0314	126.870 5677	136.098 9283	146.098 2135
47	100.396 5010	115.350 9726	132.945 3904	142.883 1328	153.672 6331
48	104.408 3960	120.388 2566	139.263 2060	149.955 6659	161.587 9016
49	108.540 6478	125.601 8456	145.833 7343	157.328 7817	169.859 3572
50	112.796 8673	130.997 9102	152.667 0837	165.015 2550	178.503 0283
60	163.053 4368	196.516 8829	237.990 6852	262.344 7398	289.497 9540
70	230.594 0637	288.937 8646	364.290 4588	409.917 1129	461.869 6796
80	321.363 0186	419.306 7868	551.244 9768	633.668 4800	729.557 6985
90	443.348 9036	603.205 0270	827.983 3335	972.923 5402	1145.269 0066
100	607.287 7327	862.611 6567	1237.623 7046	1487.306 9707	1790.855 9563

TABLE III. AMOUNT OF 1 PER ANNUM—*Continued*

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	4¾ per cent	5 per cent	6 per cent	7 per cent	8 per cent
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000
2	2.047 5000	2.050 0000	2.060 0000	2.070 0000	2.080 0000
3	3.144 7562	3.152 5000	3.183 6000	3.214 9000	3.246 4000
4	4.294 1322	4.310 1250	4.374 6160	4.439 9430	4.506 1120
5	5.498 1034	5.525 6312	5.637 0930	5.750 7390	5.866 6010
6	6.759 2634	6.801 9128	6.975 3185	7.153 2907	7.335 9290
7	8.080 3284	8.142 0084	8.393 8376	8.654 0211	8.922 8034
8	9.464 1440	9.549 1089	9.897 4679	10.259 8026	10.636 6276
9	10.913 6908	11.026 5643	11.491 3160	11.977 9888	12.487 5578
10	12.432 0911	12.577 8925	13.180 7949	13.816 4480	14.486 5625
11	14.022 6154	14.206 7872	14.971 6426	15.7835 993	16.645 4875
12	15.688 6897	15.917 1265	16.869 9412	17.888 4513	18.977 1265
13	17.433 9024	17.712 9828	18.882 1377	20.140 6429	21.495 2966
14	19.262 0128	19.598 6320	21.015 0659	22.550 4879	24.214 9203
15	21.176 9584	21.578 5636	23.275 9699	25.129 0220	27.152 1139
16	23.182 8640	23.657 4918	25.672 5281	27.888 0536	30.324 2830
17	25.284 0500	25.840 3664	28.212 8798	30.840 2173	33.750 2257
18	27.485 0424	28.132 3847	30.905 6526	33.999 0325	37.450 2437
19	29.790 5819	30.539 0039	33.759 9917	37.378 9648	41.446 2632
20	32.205 6345	33.065 9541	36.785 5912	40.995 4923	45.761 9643
21	34.735 4022	35.719 2518	39.992 7267	44.865 1768	50.422 9214
22	37.385 3338	38.505 2144	43.392 2903	49.005 7392	55.456 7552
23	40.161 1371	41.430 4751	46.995 8277	53.436 1409	60.893 2956
24	43.068 7911	44.501 9989	50.815 5774	58.176 6708	66.764 7592
25	46.114 5587	47.727 0988	54.864 5120	63.249 0377	73.105 9400
26	49.305 0002	51.113 4538	59.156 3827	68.676 4704	79.954 4152
27	52.646 9877	54.669 1264	63.705 7657	74.483 8233	87.350 7684
28	56.147 7197	58.402 5828	68.528 1116	80.697 6909	95.338 8298
29	59.814 7363	62.322 7119	73.639 7983	87.346 5293	103.965 9362
30	63.655 9363	66.438 8475	79.058 1862	94.460 7863	113.283 2111
31	67.679 5933	70.760 7899	84.801 6774	102.073 0414	123.345 8680
32	71.894 3740	75.298 8294	90.889 7780	110.218 1543	134.213 5374
33	76.309 3567	80.063 7708	97.343 1647	118.933 4251	145.950 6204
34	80.934 0512	85.066 9594	104.183 7546	128.258 7648	158.626 6701
35	85.778 4186	90.320 3074	111.434 7799	138.236 8784	172.316 8037
36	90.852 8935	95.836 3227	119.120 8667	148.913 4598	187.102 1480
37	96.168 4059	101.628 1389	127.268 1187	160.337 4020	203.070 3198
38	101.736 4052	107.709 5458	135.904 2058	172.561 0202	220.315 9454
39	107.568 8845	114.095 0231	145.058 4581	185.640 2916	238.941 2210
40	113.678 4065	120.799 7742	154.761 9656	199.635 1120	259.056 5187
41	120.078 1308	127.839 7630	165.047 6836	214.609 5698	280.781 0402
42	126.781 8420	135.231 7511	175.950 5446	230.632 2397	304.243 5234
43	133.803 9795	142.993 3387	187.507 5772	247.776 4965	329.583 0053
44	141.159 6685	151.143 0056	199.758 0319	266.120 8512	356.949 6457
45	148.864 7528	159.700 1559	212.743 5138	285.749 3108	386.505 6174
46	156.935 8285	168.685 1637	226.508 1246	306.751 7626	418.426 0668
47	165.390 2804	178.119 4218	241.098 6121	329.224 3860	452.900 1521
48	174.246 3187	188.025 3929	256.564 5288	353.270 0930	490.132 1643
49	183.523 0188	198.426 6626	272.958 4006	378.998 9995	530.342 7374
50	193.240 3622	209.347 9957	290.335 9046	406.528 9295	573.770 1564
60	319.785 5885	353.583 7179	533.128 1809	813.520 3834	1253.213 2958
70	521.058 8495	588.528 5107	967.932 1696	1614.134 1742	2720.080 0738
80	841.188 8678	971.228 8213	1746.599 8914	3189.062 6797	5886.935 4283
90	1350.363 4500	1594.607 3010	3141.075 1872	6287.185 4268	12723.938 6160
100	2160.218 0106	2610.025 1569	5638.368 0586	12381.661 7938	27484.515 7043

TABLE IV. PRESENT VALUE OF 1 PER ANNUM

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

n	½ per cent	1 per cent	1¼ per cent	1½ per cent	1¾ per cent
1	0.995 0249	0.990 0990	0.987 6543	0.985 2217	0.982 8010
2	1.985 0994	1.970 3951	1.963 1154	1.955 8834	1.948 6988
3	2.970 2481	2.940 9852	2.926 5337	2.912 2004	2.897 9840
4	3.950 4957	3.901 9656	3.878 0580	3.854 3846	3.830 9425
5	4.925 8663	4.853 4312	4.817 8350	4.782 6450	4.747 8551
6	5.896 3844	5.795 4765	5.746 0099	5.697 1872	5.648 9976
7	6.862 0740	6.728 1945	6.662 7258	6.598 2140	6.534 6414
8	7.822 9592	7.651 6778	7.568 1243	7.485 9251	7.405 0530
9	8.779 0639	8.566 0176	8.462 3450	8.360 5173	8.260 4943
10	9.730 4119	9.471 3045	9.345 5259	9.222 1846	9.101 2229
11	10.677 0267	10.367 6282	10.217 8034	10.071 1178	9.927 4918
12	11.618 9321	11.255 0775	11.079 3120	10.907 5052	10.739 5497
13	12.556 1513	12.133 7401	11.930 1847	11.731 5322	11.537 6410
14	13.488 7078	13.003 7030	12.770 5528	12.543 3815	12.322 0059
15	14.416 6246	13.865 0525	13.600 5459	13.343 2330	13.092 8805
16	15.339 9250	14.717 8738	14.420 2923	14.131 2640	13.850 4968
17	16.258 6319	15.562 2513	15.229 9183	14.907 6493	14.595 0828
18	17.172 7680	16.398 2686	16.029 5489	15.672 5609	15.326 8627
19	18.082 3562	17.226 0085	16.819 3076	16.426 1684	16.046 0567
20	18.987 4192	18.045 5530	17.599 3161	17.168 6388	16.752 8813
21	19.887 9792	18.856 9831	18.369 6950	17.900 1367	17.447 5492
22	20.784 0590	19.660 3793	19.130 5629	18.620 8244	18.130 2695
23	21.675 6806	20.455 8211	19.882 0374	19.330 8614	18.801 2476
24	22.562 8662	21.243 3873	20.624 2345	20.030 4054	19.460 6856
25	23.445 6380	22.023 1557	21.357 2686	20.719 6112	20.108 7820
26	24.324 0179	22.795 2037	22.081 2530	21.398 6317	20.745 7317
27	25.198 0278	23.559 6076	22.796 2992	22.067 6175	21.371 7264
28	26.067 6894	24.316 4432	23.502 5178	22.726 7167	21.986 9547
29	26.933 0242	25.065 7853	24.200 0176	23.376 0756	22.591 6017
30	27.794 0540	25.807 7082	24.888 9062	24.015 8380	23.185 8493
31	28.650 8000	26.542 2854	25.569 2901	24.646 1458	23.769 8765
32	29.503 2836	27.269 5895	26.241 2742	25.267 1387	24.343 8590
33	30.351 5259	27.989 6926	26.904 9622	25.878 9544	24.907 9695
34	31.195 5482	28.702 6659	27.560 4564	26.481 7285	25.462 3779
35	32.035 3713	29.408 5801	28.207 8582	27.075 5946	26.007 2510
36	32.871 0162	30.107 5050	28.847 2674	27.660 6843	26.542 7528
37	33.702 5037	30.799 5099	29.478 7826	28.237 1274	27.069 0446
38	34.529 8544	31.484 6633	30.102 5013	28.805 0516	27.586 2846
39	35.353 0890	32.163 0330	30.718 5198	29.364 5829	28.094 6286
40	36.172 2279	32.834 6861	31.326 9332	29.915 8452	28.594 2296
41	36.987 2914	33.499 6892	31.927 8352	30.458 9608	29.085 2379
42	37.798 2999	34.158 1081	32.521 3187	30.994 0500	29.567 8014
43	38.605 2735	34.810 0081	33.107 4753	31.521 2316	30.042 0652
44	39.408 2324	35.455 4535	33.686 3954	32.040 6222	30.508 1722
45	40.207 1964	36.094 5084	34.258 1682	32.552 3372	30.966 2626
46	41.002 1855	36.727 2361	34.822 8822	33.056 4898	31.416 4743
47	41.793 2194	37.353 6991	35.380 6244	33.553 1920	31.858 9428
48	42.580 3178	37.973 9595	35.931 4809	34.042 5536	32.293 8013
49	43.363 5003	38.588 0787	36.475 5367	34.524 6834	32.721 1806
50	44.142 7864	39.196 1175	37.012 8757	34.999 6881	33.141 2095
60	51.725 5608	44.955 0384	42.034 5918	39.380 2689	36.963 9855
70	58.939 4176	50.168 5144	46.469 6756	43.154 8718	40.177 9027
80	65.802 3054	54.888 2061	50.386 6571	46.407 3235	42.879 9347
90	72.331 2996	59.160 8815	53.846 0604	49.209 8545	45.151 6104
100	78.542 6448	63.028 8788	56.901 3394	51.624 7037	47.061 4730

TABLE IV. PRESENT VALUE OF 1 PER ANNUM—*Continued*

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

n	2 per cent	$2\frac{1}{8}$ per cent	$2\frac{1}{4}$ per cent	$2\frac{3}{8}$ per cent	$2\frac{1}{2}$ per cent
1	0.980 3922	0.979 1922	0.977 9951	0.97 68010	0.975 6098
2	1.941 5609	1.938 0095	1.934 4696	1.93 09411	1.927 4242
3	2.883 8833	2.876 8758	2.869 8969	2.86 29462	2.856 0236
4	3.807 7287	3.796 2065	3.784 7402	3.77 33296	3.761 9742
5	4.713 4595	4.696 4078	4.679 4525	4.66 25930	4.645 8285
6	5.601 4309	5.577 8779	5.554 4768	5.531 2264	5.508 1254
7	6.471 9911	6.441 0065	6.410 2463	6.379 7083	6.349 3906
8	7.325 4814	7.286 1753	7.247 1846	7.208 5063	7.170 1372
9	8.162 2367	8.113 7579	8.065 7062	8.018 0769	7.970 8655
10	8.982 5850	8.924 1204	8.866 2164	8.808 8664	8.752 0639
11	9.786 8480	9.717 6209	9.649 1113	9.581 3102	9.514 2087
12	10.575 3412	10.494 6104	10.414 7788	10.335 8342	10.257 7646
13	11.348 3738	11.255 4325	11.163 5979	11.072 8539	10.983 1850
14	12.106 2488	12.000 4235	11.895 9392	11.792 7755	11.690 9122
15	12.849 2635	12.729 9129	12.612 1655	12.495 9956	12.381 3777
16	13.577 7093	13.444 2231	13.312 6313	13.182 9017	13.055 0027
17	14.291 8719	14.143 6701	13.997 6834	13.853 8722	13.712 1977
18	14.992 0312	14.828 5632	14.667 6611	14.509 2769	14.353 3636
19	15.678 4620	15.499 2051	15.322 8959	15.149 4768	14.978 8913
20	16.351 4333	16.155 8923	15.963 7124	15.774 8247	15.589 1623
21	17.011 2092	16.798 9154	16.590 4278	16.385 6652	16.184 5486
22	17.658 0482	17.428 5585	17.203 3523	16.982 3347	16.765 4132
23	18.292 2041	18.045 1002	17.802 7896	17.565 1621	17.332 1105
24	18.913 9256	18.648 8129	18.389 0362	18.134 4685	17.884 9858
25	19.523 4565	19.239 9636	18.962 3826	18.690 5675	18.424 3764
26	20.121 0358	19.818 8138	19.523 1126	19.233 7656	18.950 6111
27	20.706 8978	20.385 6194	20.071 5038	19.764 3620	19.464 0109
28	21.281 2724	20.940 6310	20.607 8276	20.282 6491	19.964 8887
29	21.844 3847	21.484 0940	21.132 3498	20.788 9124	20.453 5499
30	22.396 4556	22.016 2487	21.645 3298	21.283 4309	20.930 2926
31	22.937 7015	22.537 3305	22.147 0219	21.766 4771	21.395 4074
32	23.468 3348	23.047 5696	22.637 6742	22.238 3171	21.849 1780
33	23.988 5636	23.547 1918	23.117 5298	22.699 2108	22.291 8809
34	24.498 5917	24.036 4179	23.586 8262	23.149 4123	22.723 7863
35	24.998 6193	24.515 4643	24.045 7958	23.589 1695	23.145 1573
36	25.488 8425	24.984 5428	24.494 6658	24.018 7248	23.556 2511
37	25.969 4534	25.443 8607	24.933 6585	24.438 3148	23.957 3181
38	26.440 6406	25.893 6213	25.362 9912	24.848 1707	24.348 6030
39	26.902 5888	26.334 0233	25.782 8765	25.248 5184	24.730 3444
40	27.354 3792	26.765 2615	26.193 5222	25.639 5784	25.102 7750
41	27.799 4894	27.187 5265	26.595 1317	26.021 5662	25.466 1220
42	28.234 7936	27.601 0052	26.987 9039	26.394 6923	25.820 6068
43	28.661 5623	28.005 8802	27.372 0332	26.759 1622	26.166 4457
44	29.079 9631	28.402 3307	27.747 7097	27.115 1768	26.503 8494
45	29.490 1599	28.790 5319	28.115 1195	27.462 9321	26.833 0239
46	29.892 3136	29.170 6555	28.474 4445	27.802 6199	27.154 1696
47	30.286 5820	29.542 8695	28.825 8626	28.134 4272	27.467 4826
48	30.673 1196	29.907 3385	29.169 5478	28.458 5370	27.773 1537
49	31.052 0780	30.264 2238	29.505 6702	28.775 1277	28.071 3695
50	31.423 6059	30.613 6830	29.834 3963	29.084 3738	28.362 3117
60	34.760 8867	33.732 2993	32.748 9528	31.808 4816	30.908 6565
70	37.498 6193	36.259 5088	35.082 0849	33.962 6772	32.897 8570
80	39.744 5136	38.307 4645	36.949 7808	35.666 1923	34.451 8172
90	41.586 9292	39.967 0508	38.444 8902	37.013 3140	35.665 7685
100	43.098 3516	41.311 9173	39.641 7405	38.078 6036	36.614 1053

TABLE IV. PRESENT VALUE OF 1 PER ANNUM—*Continued*

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

n	3 per cent	3½ per cent	4 per cent	4¼ per cent	4½ per cent
1	0.970 8738	0.966 1836	0.961 5385	0.959 2326	0.956 9378
2	1.913 4697	1.899 6943	1.886 0947	1.879 3598	1.872 6678
3	2.828 6114	2.801 6370	2.775 0910	2.761 9758	2.748 9644
4	3.717 0984	3.673 0792	3.629 8952	3.608 6099	3.587 5257
5	4.579 7072	4.515 0524	4.451 8223	4.420 7290	4.389 9767
6	5.417 1914	5.328 5530	5.242 1369	5.199 7400	5.157 8725
7	6.230 2830	6.114 5440	6.002 0547	5.946 9928	5.892 7009
8	7.019 6922	6.873 9555	6.732 7449	6.663 7821	6.595 8861
9	7.786 1089	7.607 6865	7.435 3316	7.351 3497	7.268 7905
10	8.530 2028	8.316 6053	8.110 8958	8.010 8870	7.912 7182
11	9.252 624	9.001 5510	8.760 4767	8.643 5367	8.528 9169
12	9.954 0040	9.663 3343	9.385 0738	9.250 3949	9.118 5808
13	10.634 9553	10.302 7385	9.985 6478	9.832 5131	9.682 8524
14	11.296 0731	10.920 5203	10.563 1229	10.390 8999	10.222 8253
15	11.937 9351	11.517 4109	11.118 3874	10.926 5226	10.739 5457
16	12.561 1020	12.094 1168	11.652 2956	11.440 3095	11.234 0150
17	13.166 1185	12.651 3206	12.165 6688	11.933 1506	11.707 1914
18	13.753 5131	13.189 6817	12.659 2970	12.405 8998	12.159 9918
19	14.323 7991	13.709 8374	13.133 9394	12.859 3764	12.593 2936
20	14.877 4749	14.212 4033	13.590 3263	13.294 3658	13.007 9364
21	15.415 0241	14.697 9742	14.029 1600	13.711 6219	13.404 7239
22	15.936 9166	15.167 1248	14.451 1153	14.111 8675	13.784 4248
23	16.443 6084	15.620 4105	14.856 8417	14.495 7962	14.147 7749
24	16.935 5421	16.058 3676	15.246 9631	14.864 0731	14.495 4784
25	17.413 1477	16.481 5146	15.622 0799	15.217 3363	14.828 2090
26	17.876 8424	16.890 3523	15.982 7692	15.556 1979	15.146 6114
27	18.327 0315	17.285 3645	16.329 5858	15.881 2450	15.451 3028
28	18.764 1082	17.667 0188	16.663 0632	16.193 0407	15.742 8735
29	19.188 4546	18.035 7670	16.983 7146	16.492 1254	16.021 8885
30	19.600 4414	18.392 0454	17.292 0333	16.779 0172	16.288 8885
31	20.000 4285	18.736 2758	17.588 4936	17.054 2131	16.544 3910
32	20.388 7655	19.068 8655	17.873 5515	17.318 1900	16.788 8909
33	20.765 7918	19.390 2082	18.147 6457	17.571 4053	17.022 8621
34	21.131 8367	19.700 6842	18.411 1978	17.814 2977	17.246 7580
35	21.487 2201	20.000 6611	18.664 6132	18.047 2879	17.461 0124
36	21.832 2525	20.290 4938	18.908 2820	18.270 7798	17.666 0406
37	22.167 2354	20.570 5254	19.142 5788	18.485 1605	17.862 2398
38	22.492 4616	20.841 0874	19.367 8642	18.690 8014	18.049 9902
39	22.808 2151	21.102 4999	19.584 4848	18.888 0589	18.229 6557
40	23.114 7720	21.355 0723	19.792 7739	19.077 2747	18.401 5844
41	23.412 4000	21.599 1037	19.993 0518	19.258 7767	18.566 1095
42	23.701 3592	21.834 8828	20.185 6267	19.432 8793	18.723 5498
43	23.981 9021	22.062 6887	20.370 7949	19.599 8843	18.874 2103
44	24.254 2739	22.282 7910	20.548 8413	19.760 0808	19.018 3830
45	24.518 7125	22.495 4503	20.720 0397	19.913 7466	19.156 3474
46	24.775 4491	22.700 9181	20.884 6536	20.061 1478	19.288 3707
47	25.024 7078	22.899 4378	21.042 9361	20.202 5399	19.414 7088
48	25.266 7066	23.091 2442	21.195 1309	20.338 1677	19.535 6065
49	25.501 6569	23.276 5645	21.341 4720	20.468 2664	19.651 2981
50	25.729 7640	23.455 6179	21.482 1846	20.593 0613	19.762 0078
60	27.675 5637	24.944 7341	22.623 4900	21.592 7791	20.638 0220
70	29.123 4214	26.000 3966	23.394 5150	22.252 1303	21.202 1119
80	30.200 7634	26.748 7757	23.915 3918	22.686 9970	21.565 3449
90	31.002 4071	27.279 3156	24.267 2776	22.973 8078	21.799 2408
100	31.598 9053	27.655 4254	24.504 9990	23.162 9702	21.949 8527

TABLE IV. PRESENT VALUE OF 1 PER ANNUM—*Continued*

$$a_{\overline{n}|} = \frac{1 - v^n}{i}$$

n	$4\frac{3}{4}$ per cent	5 per cent	6 per cent	7 per cent	8 per cent
1	0.954 6539	0.952 3810	0.943 3962	0.934 5794	0.925 9259
2	1.866 0181	1.859 4104	1.833 3927	1.808 0182	1.783 2648
3	2.736 0554	2.723 2480	2.673 0120	2.624 3160	2.577 0970
4	3.566 6400	3.545 9505	3.465 1056	3.387 2113	3.312 1268
5	4.359 5609	4.329 4767	4.212 3638	4.100 1974	3.992 7100
6	5.116 5259	5.075 6921	4.917 3243	4.766 5397	4.622 8797
7	5.839 1656	5.786 3734	5.582 3814	5.389 2894	5.206 3701
8	6.529 0363	6.463 2128	6.209 7938	5.971 2985	5.746 6389
9	7.187 6242	7.107 8217	6.801 6923	6.515 2322	6.246 8879
10	7.816 3477	7.721 7349	7.360 0870	7.023 5816	6.710 0814
11	8.416 5610	8.306 4142	7.886 8746	7.498 6744	7.138 9643
12	8.989 5571	8.863 2516	8.383 8439	7.942 6863	7.536 0780
13	9.536 5700	9.393 5730	8.852 6830	8.357 6508	7.903 7759
14	10.058 7780	9.898 6409	9.294 9839	8.745 4680	8.244 2370
15	10.557 3060	10.379 6580	9.712 2490	9.107 9140	8.559 4787
16	11.033 2277	10.837 7696	10.105 8953	9.446 6486	8.851 3692
17	11.487 5682	11.274 0662	10.477 2597	9.763 2230	9.121 6381
18	11.921 3062	11.689 5869	10.827 6035	10.059 0869	9.371 8871
19	12.335 3758	12.085 3209	11.158 1165	10.335 5952	9.603 5992
20	12.730 6690	12.462 2103	11.469 9212	10.594 0143	9.818 1474
21	13.108 0372	12.821 1527	11.764 0766	10.835 5273	10.016 8032
22	13.468 2933	13.163 0026	12.041 5817	11.061 2405	10.200 7437
23	13.812 2132	13.488 5739	12.303 3790	11.272 1874	10.371 0590
24	14.140 5376	13.798 6418	12.550 3575	11.469 3340	10.528 7583
25	14.453 9739	14.093 9446	12.783 3562	11.653 5832	10.674 7762
26	14.753 1970	14.375 1853	13.003 1662	11.825 7787	10.809 9780
27	15.038 8516	14.643 0336	13.210 5341	11.986 7090	10.935 1648
28	15.311 5528	14.898 1273	13.406 1643	12.137 1113	11.051 0785
29	15.571 8881	15.141 0736	13.590 7210	12.277 6741	11.158 4060
30	15.820 4183	15.372 4510	13.764 8312	12.409 0412	11.257 7833
31	16.057 6785	15.592 8105	13.929 0860	12.531 8142	11.349 7994
32	16.284 1800	15.802 6767	14.084 0434	12.646 5553	11.434 9994
33	16.500 4105	16.002 5492	14.230 2296	12.753 7900	11.513 8884
34	16.706 8358	16.192 9040	14.368 1411	12.854 0094	11.586 9337
35	16.903 9005	16.374 1943	14.498 2446	12.947 6723	11.654 5682
36	17.092 0291	16.546 8517	14.620 9871	13.035 2078	11.717 1928
37	17.271 6269	16.711 2873	14.736 7803	13.117 0166	11.775 1785
38	17.443 0805	16.867 8927	14.846 0192	13.193 4735	11.828 8690
39	17.606 7595	17.017 0407	14.949 0747	13.264 9285	11.878 5824
40	17.763 0162	17.159 0864	15.046 2969	13.331 7088	11.924 6133
41	17.912 1873	17.294 3680	15.138 0159	13.394 1204	11.967 2346
42	18.054 5941	17.423 2076	15.224 5433	13.452 4490	12.006 6987
43	18.190 5433	17.545 9120	15.306 1729	13.506 9617	12.043 2395
44	18.320 3277	17.662 7733	15.383 1820	13.557 9081	12.077 0736
45	18.444 2269	17.774 0698	15.455 8321	13.605 5216	12.108 4015
46	18.562 5078	17.880 0665	15.524 3699	13.650 0202	12.137 4088
47	18.675 4251	17.981 0157	15.589 0282	13.691 6076	12.164 2674
48	18.783 2221	18.077 1578	15.650 0266	13.730 4744	12.189 1365
49	18.886 1308	18.168 7217	15.707 5723	13.766 7986	12.212 1634
50	18.984 3731	18.255 9255	15.761 8606	13.800 7463	12.233 4846
60	19.752 2689	18.929 2895	16.161 4277	14.039 1812	12.376 5518
70	20.235 0630	19.342 6766	16.384 5439	14.160 3893	12.442 8196
80	20.538 6070	19.596 4605	16.509 1308	14.222 0054	12.473 5144
90	20.729 4523	19.752 2617	16.578 6994	14.253 3279	12.487 7320
100	20.849 4412	19.847 9102	16.617 5462	14.269 2507	12.494 3176

TABLE V. ANNUITY WHICH 1 WILL BUY

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	½ per cent	1 per cent	1¼ per cent	1½ per cent	1¾ per cent
1	1.005 0000	1.010 0000	1.012 5000	1.015 0000	1.017 5000
2	0.503 7531	0.507 5124	0.509 3944	0.511 2779	0.513 1630
3	0.336 6722	0.340 0221	0.341 7012	0.343 3830	0.345 0675
4	0.253 1328	0.256 2811	0.257 8610	0.259 4448	0.261 0324
5	0.203 0100	0.206 0398	0.207 5621	0.209 0893	0.210 6214
6	0.169 5955	0.172 5484	0.174 0338	0.175 5252	0.177 0226
7	0.145 7285	0.148 6283	0.150 0887	0.151 5562	0.153 0306
8	0.127 8289	0.130 6903	0.132 1331	0.133 5840	0.135 0429
9	0.113 9074	0.116 7404	0.118 1706	0.119 6098	0.121 0581
10	0.102 7706	0.105 5821	0.107 0031	0.108 4342	0.109 8754
11	0.093 6590	0.096 4541	0.097 8684	0.099 2938	0.100 7304
12	0.086 0664	0.088 8488	0.090 2583	0.091 6800	0.093 1138
13	0.079 6422	0.082 4148	0.083 8210	0.085 2404	0.086 6728
14	0.074 1361	0.076 9012	0.078 3052	0.079 7233	0.081 1556
15	0.069 3644	0.072 1238	0.073 5265	0.074 9444	0.076 3774
16	0.065 1894	0.067 9446	0.069 3467	0.070 7651	0.072 1996
17	0.061 5058	0.064 2581	0.065 6602	0.067 0796	0.068 5162
18	0.058 2317	0.060 9820	0.062 3848	0.063 8058	0.065 2449
19	0.055 3025	0.058 0518	0.059 4555	0.060 8785	0.062 3206
20	0.052 6664	0.055 4153	0.056 8204	0.058 2457	0.059 6912
21	0.050 2816	0.053 0308	0.054 4375	0.055 8655	0.057 3146
22	0.048 1138	0.050 8637	0.052 2724	0.053 7033	0.055 1564
23	0.046 1346	0.048 8858	0.050 2967	0.051 7308	0.053 1880
24	0.044 3206	0.047 0735	0.048 4866	0.049 9241	0.051 3856
25	0.042 6519	0.045 4068	0.046 8225	0.048 2634	0.049 7295
26	0.041 1116	0.043 8689	0.045 2873	0.046 7320	0.048 2021
27	0.039 6856	0.042 4455	0.043 8668	0.045 3153	0.046 7908
28	0.038 3617	0.041 1244	0.042 5486	0.044 0011	0.045 4815
29	0.037 1291	0.039 8950	0.041 3223	0.042 7788	0.044 2642
30	0.035 9789	0.038 7481	0.040 1785	0.041 6392	0.043 1298
31	0.034 9030	0.037 6757	0.039 1094	0.040 5743	0.042 0700
32	0.033 8945	0.036 6709	0.038 1079	0.039 5771	0.041 0781
33	0.032 9473	0.035 7274	0.037 1679	0.038 6414	0.040 1478
34	0.032 0559	0.034 8400	0.036 2839	0.037 7619	0.039 2736
35	0.031 2155	0.034 0037	0.035 4511	0.036 9336	0.038 4508
36	0.030 4219	0.033 2143	0.034 6653	0.036 1524	0.037 6751
37	0.029 6714	0.032 4680	0.033 9227	0.035 4144	0.036 9426
38	0.028 9604	0.031 7615	0.033 2198	0.034 7161	0.036 2499
39	0.028 2861	0.031 0916	0.032 5536	0.034 0546	0.035 5940
40	0.027 6455	0.030 4556	0.031 9214	0.033 4271	0.034 9721
41	0.027 0363	0.029 8510	0.031 3206	0.032 8311	0.034 3817
42	0.026 4562	0.029 2756	0.030 7491	0.032 2643	0.033 8206
43	0.025 9032	0.028 7274	0.030 2047	0.031 7246	0.033 2867
44	0.025 3754	0.028 2044	0.029 6856	0.031 2104	0.032 7781
45	0.024 8712	0.027 7050	0.029 1901	0.030 7198	0.032 2932
46	0.024 3889	0.027 2278	0.028 7168	0.030 2512	0.031 8304
47	0.023 9273	0.026 7711	0.028 2641	0.029 8034	0.031 3884
48	0.023 4850	0.026 3338	0.027 8307	0.029 3750	0.030 9657
49	0.023 0609	0.025 9147	0.027 4156	0.028 9648	0.030 5612
50	0.022 6538	0.025 5127	0.027 0176	0.028 5717	0.030 1739
60	0.019 3328	0.022 2444	0.023 7899	0.025 3934	0.027 0534
70	0.016 9666	0.019 9328	0.021 5194	0.023 1724	0.024 8893
80	0.015 1970	0.018 2188	0.019 8465	0.021 5483	0.023 3209
90	0.013 8253	0.016 9031	0.018 5715	0.020 3211	0.022 1476
100	0.012 7319	0.015 8657	0.017 5743	0.019 3706	0.021 2488

TABLE V. ANNUITY WHICH 1 WILL BUY—*Continued*

$$\frac{1}{a_{\overline{n|}}} = \frac{1}{s_{\overline{n|}}} + i$$

n	2 per cent	2 1/8 per cent	2¼ per cent	2 3/8 per cent	2½ per cent
1	1.020 0000	1.021 2500	1.022 5000	1.023 7500	1.025 0000
2	0.515 0495	0.515 9934	0.516 9376	0.517 8822	0.518 8272
3	0.346 7547	0.347 5993	0.348 4446	0.349 2905	0.350 1372
4	0.262 6238	0.263 4209	0.264 2189	0.265 0179	0.265 8179
5	0.212 1584	0.212 9287	0.213 7002	0.214 4729	0.215 2469
6	0.178 5258	0.179 2796	0.180 0350	0.180 7917	0.181 5500
7	0.154 5120	0.155 2552	0.156 0002	0.156 7470	0.157 4954
8	0.136 5098	0.137 2462	0.137 9846	0.138 7250	0.139 4674
9	0.122 5154	0.123 2474	0.123 9817	0.124 7182	0.125 4569
10	0.111 3265	0.112 0559	0.112 7877	0.113 5220	0.114 2588
11	0.102 1779	0.102 9058	0.103 6365	0.104 3699	0.105 1060
12	0.094 5596	0.095 2870	0.096 0174	0.096 7508	0.097 4871
13	0.088 1184	0.088 8460	0.089 5769	0.090 3110	0.091 0483
14	0.082 6020	0.083 3304	0.084 0623	0.084 7977	0.085 5365
15	0.077 8255	0.078 5551	0.079 2885	0.080 0256	0.080 7665
16	0.073 6501	0.074 3814	0.075 1166	0.075 8558	0.076 5990
17	0.069 9698	0.070 7030	0.071 4404	0.072 1820	0.072 9278
18	0.066 7021	0.067 4374	0.068 1772	0.068 9214	0.069 6701
19	0.063 7818	0.064 5194	0.065 2618	0.066 0089	0.066 7606
20	0.061 1567	0.061 8969	0.062 6421	0.063 3922	0.064 1471
21	0.058 7848	0.059 5277	0.060 2757	0.061 0290	0.061 7873
22	0.056 6314	0.057 3771	0.058 1282	0.058 8847	0.059 6466
23	0.054 6681	0.055 4167	0.056 1710	0.056 9309	0.057 6964
24	0.052 8711	0.053 6227	0.054 3802	0.055 1436	0.055 9128
25	0.051 2204	0.051 9752	0.052 7360	0.053 5029	0.054 2759
26	0.049 6992	0.050 4571	0.051 2213	0.051 9919	0.052 7688
27	0.048 2931	0.049 0542	0.049 8219	0.050 5961	0.051 3769
28	0.046 9897	0.047 7540	0.048 5253	0.049 3032	0.050 0879
29	0.045 7784	0.046 5461	0.047 3208	0.048 1026	0.048 8913
30	0.044 6499	0.045 4210	0.046 1993	0.046 9849	0.047 7776
31	0.043 5964	0.044 3708	0.045 1528	0.045 9422	0.046 7390
32	0.042 6106	0.043 3885	0.044 1742	0.044 9674	0.045 7683
33	0.041 6865	0.042 4679	0.043 2572	0.044 0544	0.044 8594
34	0.040 8187	0.041 6035	0.042 3966	0.043 1976	0.044 0068
35	0.040 0022	0.040 7906	0.041 5873	0.042 3923	0.043 2056
36	0.039 2328	0.040 0248	0.040 8252	0.041 6342	0.042 4516
37	0.038 5068	0.039 3022	0.040 1064	0.040 9194	0.041 7409
38	0.037 8206	0.038 6196	0.039 4275	0.040 2444	0.041 0701
39	0.037 1711	0.037 9737	0.038 7854	0.039 6063	0.040 4362
40	0.036 5558	0.037 3619	0.038 1774	0.039 0022	0.039 8362
41	0.035 9719	0.036 7816	0.037 6009	0.038 4297	0.039 2679
42	0.035 4173	0.036 2306	0.037 0536	0.037 8864	0.038 7288
43	0.034 8899	0.035 7068	0.036 5336	0.037 3704	0.038 2169
44	0.034 3879	0.035 2084	0.036 0390	0.036 8797	0.037 7304
45	0.033 9096	0.034 7336	0.035 5680	0.036 4127	0.037 2675
46	0.033 4534	0.034 2810	0.035 1192	0.035 9678	0.036 8268
47	0.033 0179	0.033 8491	0.034 6911	0.035 5436	0.036 4067
48	0.032 6018	0.033 4366	0.034 2823	0.035 1388	0.036 0060
49	0.032 2040	0.033 0423	0.033 8918	0.034 7522	0.035 6235
50	0.031 8232	0.032 6651	0.033 5184	0.034 3827	0.035 2581
60	0.028 7680	0.029 6452	0.030 5353	0.031 4382	0.032 3534
70	0.026 6676	0.027 5790	0.028 5046	0.029 4441	0.030 3971
80	0.025 1607	0.026 1046	0.027 0638	0.028 0378	0.029 0260
90	0.024 0460	0.025 0206	0.026 0113	0.027 0173	0.028 0381
100	0.023 2027	0.024 2061	0.025 2259	0.026 2615	0.027 3119

TABLE V. ANNUITY WHICH 1 WILL BUY—*Continued*

$$\frac{1}{a_{\overline{n|}}} = \frac{1}{s_{\overline{n|}}} + i$$

n	3 per cent	3½ per cent	4 per cent	4¼ per cent	4½ per cent
1	1.030 0000	1.035 0000	1.040 0000	1.042 5000	1.045 0000
2	0.522 6108	0.526 4005	0.530 1961	0.532 0961	0.533 9976
3	0.353 5304	0.356 9342	0.360 3485	0.362 0596	0.363 7734
4	0.269 0270	0.272 2511	0.275 4900	0.277 1150	0.278 7436
5	0.218 3546	0.221 4814	0.224 6271	0.226 2070	0.227 7916
6	0.184 5975	0.187 6682	0.190 7619	0.192 3173	0.193 8784
7	0.160 5064	0.163 5445	0.166 6096	0.168 1522	0.169 7015
8	0.142 4564	0.145 4766	0.148 5278	0.150 0649	0.151 6096
9	0.128 4339	0.131 4460	0.134 4930	0.136 0294	0.137 5745
10	0.117 2305	0.120 2414	0.123 2909	0.124 8301	0.126 3788
11	0.108 0774	0.111 0920	0.114 1490	0.115 6934	0.117 2482
12	0.100 4621	0.103 4840	0.106 5522	0.108 1035	0.109 6662
13	0.094 0295	0.097 0616	0.100 1437	0.101 7034	0 103 2754
14	0.088 5263	0.091 5707	0.094 6690	0.096 2381	0.097 8203
15	0.083 7666	0.086 8251	0.089 9411	0.091 5204	0.093 1138
16	0.079 6108	0.082 6848	0.085 8200	0.087 4102	0.089 0154
17	0.075 9525	0.079 0431	0.082 1985	0.083 8002	0.085 4176
18	0.072 7087	0.075 8168	0.078 9933	0.080 6068	0.082 2369
19	0.069 8139	0.072 9403	0.076 1386	0.077 7643	0.079 4073
20	0.067 2157	0.070 3611	0.073 5818	0.075 2198	0.076 8761
21	0.064 8718	0.068 0366	0.071 2801	0.072 9308	0.074 6006
22	0.062 7474	0.065 9321	0.069 1988	0.070 8623	0.072 5456
23	0.060 8139	0.064 0188	0.067 3091	0.068 9855	0.070 6825
24	0.059 0474	0.062 2728	0.065 5868	0.067 2763	0.068 9870
25	0.057 4279	0.060 6740	0.064 0120	0.065 7145	0.067 4390
26	0.055 9383	0.059 2054	0.062 5674	0.064 2831	0.066 0214
27	0.054 5642	0.057 8524	0.061 2385	0.062 9674	0.064 7195
28	0.053 2932	0.056 6026	0.060 0130	0.061 7549	0.063 5208
29	0.052 1147	0.055 4454	0.058 8799	0.060 6350	0.062 4146
30	0.051 0193	0.054 3713	0.057 8301	0.059 5982	0.061 3915
31	0.049 9989	0.053 3724	0.056 8554	0.058 6365	0.060 4434
32	0.049 0466	0.052 4415	0.055 9486	0.057 7428	0.059 5632
33	0.048 1561	0.051 5724	0.055 1036	0.056 9106	0.058 7445
34	0.047 3220	0.050 7597	0.054 3148	0.056 1347	0.057 9819
35	0.046 5393	0.049 9984	0.053 5773	0.055 4100	0.057 2704
36	0.045 8038	0.049 2842	0.052 8869	0.054 7322	0.056 6058
37	0.045 1116	0.048 6132	0.052 2396	0.054 0974	0.055 9840
38	0.044 4593	0.047 9821	0.051 6319	0.053 5023	0.055 4017
39	0.043 8438	0.047 3878	0.051 0608	0.052 9435	0.054 8557
40	0.043 2624	0.046 8273	0.050 5235	0.052 4184	0.054 3432
41	0.042 7124	0.046 2982	0.050 0174	0.051 9244	0.053 8616
42	0.042 1917	0.045 7983	0.049 5402	0.051 4592	0.053 4087
43	0.041 6981	0.045 3254	0.049 0899	0.051 0207	0.052 9824
44	0.041 2298	0.044 8777	0.048 6645	0.050 6071	0.052 5807
45	0.040 7852	0.044 4534	0.048 2625	0.050 2166	0.052 2020
46	0.040 3625	0.044 0511	0.047 8820	0.049 8476	0.051 8447
47	0.039 9605	0.043 6692	0.047 5219	0.049 4987	0.051 5073
48	0.039 5778	0.043 3065	0.047 1806	0.049 1686	0.051 1886
49	0.039 2131	0.042 9617	0.046 8571	0.048 8561	0.050 8872
50	0.038 8655	0.042 6337	0.046 5502	0.048 5600	0.050 6022
60	0.036 1330	0.040 0886	0.044 2018	0.046 3118	0.048 4543
70	0.034 3366	0.038 4610	0.042 7451	0.044 9395	0.047 1651
80	0.033 1118	0.037 3849	0.041 8141	0.044 0781	0.046 3707
90	0.032 2556	0.036 6578	0.041 2078	0.043 5278	0.045 8732
100	0.031 6467	0.036 1593	0.040 8080	0.043 1724	0.045 5584

TABLE V. ANNUITY WHICH 1 WILL BUY—*Continued*

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

n	4¾ per cent	5 per cent	6 per cent	7 per cent	8 per cent
1	1.047 5000	1.050 0000	1.060 0000	1.070 0000	1.080 0000
2	0.535 9005	0.537 8049	0.545 4369	0.553 0918	0.560 7692
3	0.365 4897	0.367 2086	0.374 1098	0.381 0517	0.388 0335
4	0.280 3759	0.282 0118	0.288 5915	0.295 2281	0.301 9208
5	0.229 3809	0.230 9748	0.237 3964	0.243 8907	0.250 4564
6	0.195 4451	0.197 0175	0.203 3626	0.209 7958	0.216 3154
7	0.171 2574	0.172 8198	0.179 1350	0.185 5532	0.192 0724
8	0.153 1620	0.154 7218	0.161 0359	0.167 4678	0.174 0148
9	0.139 1280	0.140 6901	0.147 0222	0.153 4865	0.160 0797
10	0.127 9370	0.129 5046	0.135 8680	0.142 3775	0.149 0295
11	0.118 8134	0.120 3889	0.126 7929	0.133 3569	0.140 0763
12	0.111 2402	0.112 8254	0.119 2770	0.125 9020	0.132 6950
13	0.104 8595	0.106 4558	0.112 9601	0.119 6508	0.126 5218
14	0.099 4156	0.101 0240	0.107 5849	0.114 3449	0.121 2968
16	0.094 7211	0.096 3423	0.102 9628	0.109 7946	0.116 8295
16	0.090 6353	0.092 2699	0.098 9521	0.105 8576	0.112 9769
17	0.087 0506	0.088 6991	0.095 4448	0.102 4252	0.109 6294
18	0.083 8834	0.085 5462	0.092 3565	0.099 4126	0.106 7021
19	0.081 0677	0.082 7450	0.089 6209	0.096 7530	0.104 1276
20	0.078 5505	0.080 2426	0.087 1846	0.094 3929	0.101 8522
21	0.076 2891	0.077 9961	0.085 0046	0.092 2890	0.099 8322
22	0.074 2485	0.075 9705	0.083 0456	0.090 4058	0.098 0321
23	0.072 3997	0.074 1368	0.081 2785	0.088 7139	0.096 4222
24	0.070 7187	0.072 4709	0.079 6790	0.087 1890	0.094 9780
25	0.069 1851	0.070 9525	0.078 2267	0.085 8105	0.093 6788
26	0.067 7819	0.069 5643	0.076 9044	0.084 5610	0.092 5071
27	0.066 4944	0.068 2919	0.075 6972	0.083 4257	0.091 4481
28	0.065 3102	0.067 1225	0.074 5926	0.082 3919	0.090 4889
29	0.064 2183	0.066 0455	0.073 5796	0.081 4486	0.089 6185
30	0.063 2095	0.065 0514	0.072 6489	0.080 5864	0.088 8274
31	0.062 2755	0.064 1321	0.071 7922	0.079 7969	0.088 1073
32	0.061 4093	0.063 2804	0.071 0023	0.079 0729	0.087 4508
33	0.060 6046	0.062 4900	0.070 2729	0.078 4081	0.086 8516
34	0.059 8557	0.061 7554	0.069 5984	0.077 7967	0.086 3041
36	0.059 1579	0.061 0717	0.068 9739	0.077 2340	0.085 8033
36	0.058 5068	0.060 4345	0.068 3948	0.076 7153	0.085 3447
37	0.057 8984	0.059 8398	0.067 8574	0.076 2368	0.084 9244
38	0.057 3293	0.059 2842	0.067 3581	0.075 7950	0.084 5389
39	0.056 7964	0.058 7646	0.066 8938	0.075 3868	0.084 1851
40	0.056 2968	0.058 2782	0.066 4615	0.075 0091	0.083 8602
41	0.055 8279	0.057 8223	0.066 0589	0.074 6596	0.083 5615
42	0.055 3876	0.057 3947	0.065 6834	0.074 3359	0.083 2868
43	0.054 9736	0.056 9933	0.065 3331	0.074 0359	0.083 0341
44	0.054 5842	0.056 6162	0.065 0061	0.073 7577	0.082 8015
46	0.054 2175	0.056 2617	0.064 7005	0.073 4996	0.082 5873
46	0.053 8720	0.055 9282	0.064 4148	0.073 2600	0.082 3899
47	0.053 5463	0.055 6142	0.064 1477	0.073 0374	0.082 2080
48	0.053 2390	0.055 3184	0.063 8977	0.072 8307	0.082 0403
49	0.052 9489	0.055 0396	0.063 6636	0.072 6385	0.081 8856
60	0.052 6749	0.054 7767	0.063 4443	0.072 4598	0.081 7429
60	0.050 6271	0.052 8282	0.061 8757	0.071 2292	0.080 7980
70	0.049 4192	0.051 6992	0.061 0331	0.070 6195	0.080 3676
80	0.048 6888	0.051 0296	0.060 5725	0.070 3136	0.080 1699
90	0 048 2405	0.050 6271	0.060 3184	0.070 1590	0.080 0786
100	0.047 9629	0.050 3831	0.060 1774	0.070 0808	0.080 0364

TABLE VI. AMOUNT OF 1 FOR CERTAIN PARTS OF A YEAR

$$(1 + i)^{\frac{1}{p}}$$

%	p = 2	p = 4	p = 12	%	p = 2	p = 4	p = 12
½	1.002 4969	1.001 2477	1.000 4157	3	1.014 8892	1.007 4171	1.002 4663
1	1.004 9876	1.002 4907	1.000 8295	3½	1.017 3495	1.008 6374	1.002 8709
1¼	1.006 2306	1.003 1105	1.001 0357	4	1.019 8039	1.009 8534	1.003 2737
1½	1.007 4721	1.003 7291	1.001 2415	4¼	1.021 0289	1.010 4597	1.003 4745
1¾	1.008 7121	1.004 3466	1.001 4468	4½	1.022 2524	1.011 0650	1.003 6748
2	1.009 9505	1.004 9629	1.001 6516	4¾	1.023 4745	1.011 6692	1.003 8747
2⅛	1.010 5691	1.005 2707	1.001 7538	5	1.024 6951	1.012 2722	1.004 0741
2¼	1.011 1874	1.005 5782	1.001 8559	6	1.029 5630	1.014 6738	1.004 8676
2⅜	1.011 8053	1.005 8853	1.001 9579	7	1.034 4080	1.017 0585	1.005 6541
2½	1.012 4228	1.006 1922	1.002 0598	8	1.039 2305	1.019 4265	1.006 4340

TABLE VII. VALUES OF

$$j_{(p)} = p[(1 + i)^{\frac{1}{p}} - 1]$$

%	p = 2	p = 4	p = 12	%	p = 2	p = 4	p = 12
½	0.004 9938	0.004 9907	0.004 9886	3	0.029 7783	0.029 6683	0.029 5952
1	0.009 9751	0.009 9627	0.009 9545	3½	0.034 6990	0.034 5498	0.034 4508
1¼	0.012 4612	0.012 4418	0.012 4290	4	0.039 6078	0.039 4136	0.039 2849
1½	0.014 9442	0.014 9164	0.014 8978	4¼	0.042 0578	0.041 8390	0.041 6939
1¾	0.017 4241	0.017 3863	0.017 3612	4½	0.044 5048	0.044 2600	0.044 0977
2	0.019 9010	0.019 8517	0.019 8190	4¾	0.046 9489	0.046 6766	0.046 4962
2⅛	0.021 1383	0.021 0827	0.021 0458	5	0.049 3902	0.049 0889	0.048 8895
2¼	0.022 3748	0.022 3126	0.022 2712	6	0.059 1260	0.058 6954	0.058 4106
2⅜	0.023 6106	0.023 5414	0.023 4953	7	0.068 8161	0.068 2341	0.067 8497
2½	0.024 8457	0.024 7690	0.024 7180	8	0.078 4610	0.077 7062	0.077 2084

TABLE VIII. VALUES OF

$$s_{\overline{1}|}^{(p)} = \frac{i}{j_{(p)}}$$

%	p = 2	p = 4	p = 12	%	p = 2	p = 4	p = 12
½	1.001 248	1.001 873	1.002 290	3	1.007 445	1.011 181	1.013 677
1	1.002 494	1.003 742	1.004 574	3½	1.008 675	1.013 031	1.015 942
1¼	1.003 115	1.004 675	1.005 716	4	1.009 902	1.014 877	1.018 203
1½	1.003 736	1.005 608	1.006 857	4¼	1.010 514	1.015 799	1.019 333
1¾	1.004 356	1.006 539	1.007 996	4½	1.011 126	1.016 720	1.020 461
2	1.004 975	1.007 469	1.009 134	4¾	1.011 737	1.017 640	1.021 588
2⅛	1.005 285	1.007 934	1.009 703	5	1.012 348	1.018 559	1.022 715
2¼	1.005 594	1.008 398	1.010 271	6	1.014 782	1.022 227	1.027 211
2⅜	1.005 903	1.008 863	1.010 839	7	1.017 204	1.025 880	1.031 691
2½	1.006 211	1.009 327	1.011 407	8	1.019 615	1.029 519	1.036 157

TABLE IX. AMERICAN EXPERIENCE TABLE OF MORTALITY

Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x	Age x	Number living l_x	Number of deaths d_x	Yearly probability of dying q_x	Yearly probability of living p_x
10	100,000	749	0.007 490	0.992 510	53	66,797	1,091	0.016 333	0.983 667
11	99,251	746	0.007 516	0.992 484	54	65,706	1,143	0.017 396	0.982 604
12	98,505	743	0.007 543	0.992 457	55	64,563	1,199	0.018 571	0.981 429
13	97,762	740	0.007 569	0.992 431	56	63,364	1,260	0.019 885	0.980 115
14	97,022	737	0.007 596	0.992 404	57	62,104	1,325	0.021 335	0.978 665
15	96,285	735	0.007 634	0.992 366	58	60,779	1,394	0.022 936	0.977 064
16	95,550	732	0.007 661	0.992 339	59	59,385	1,468	0.024 720	0.975 280
17	94,818	729	0.007 688	0.992 312	60	57,917	1,546	0.026 693	0.973 307
18	94,089	727	0.007 727	0.992 273	61	56,371	1,628	0.028 880	0.971 120
19	93,362	725	0.007 765	0.992 235	62	54,743	1,713	0.031 292	0.968 708
20	92,637	723	0.007 805	0.992 195	63	53,030	1,800	0.033 943	0.966 057
21	91,914	722	0.077 855	0.992 145	64	51,230	1,889	0.036 873	0.963 127
22	91,192	721	0.007 906	0.992 094	65	49,341	1,980	0.040 129	0.959 871
23	90,471	720	0.007 958	0.992 042	66	47,361	2,070	0.043 707	0.956 293
24	89,751	719	0.008 011	0.991 989	67	45,291	2,158	0.047 647	0.952 353
25	89,032	718	0.008 065	0.991 935	68	43,133	2,243	0.052 002	0.947 998
26	88,314	718	0.008 130	0.991 870	69	40,890	2,321	0.056 762	0.943 238
27	87,596	718	0.008 197	0.991 803	70	38,569	2,391	0.061 993	0.938 007
28	86,878	718	0.008 264	0.991 736	71	36,178	2,448	0.067 665	0.932 335
29	86,160	719	0.008 345	0.991 655	72	33,730	2,487	0.073 733	0.926 267
30	85,441	720	0.008 427	0.991 573	73	31,243	2,505	0.080 178	0.919 822
31	84,721	721	0.008 510	0.991 490	74	28,738	2,501	0.087 028	0.912 972
32	84,000	723	0.008 607	0.991 393	75	26,237	2,476	0.094 371	0.905 629
33	83,277	726	0.008 718	0.991 282	76	23,761	2,431	0.102 311	0.897 689
34	82,551	729	0.008 831	0.991 169	77	21,330	2,369	0.111 064	0.888 936
35	81,822	732	0.008 946	0.991 054	78	18,961	2,291	0.120 827	0.879 173
36	81,090	737	0.009 089	0.990 911	79	16,670	2,196	0.131 734	0.868 266
37	80,353	742	0.009 234	0.990 766	80	14,474	2,091	0.144 466	0.855 534
38	79,611	749	0.009 408	0.990 592	81	12,383	1,964	0.158 605	0.841 395
39	78,862	756	0.009 586	0.990 414	82	10,419	1,816	0.174 297	0.825 703
40	78,106	765	0.009 794	0.990 206	83	8,603	1,648	0.191 561	0.808 439
41	77,341	774	0.010 008	0.989 992	84	6,955	1,470	0.211 359	0.788 641
42	76,567	785	0.010 252	0.989 748	85	5,485	1,292	0.235 552	0.764 448
43	75,782	797	0.010 517	0.989 483	86	4,193	1,114	0.265 681	0.734 319
44	74,985	812	0.010 829	0.989 171	87	3,079	933	0.303 020	0.696 980
45	74,173	828	0.011 163	0.988 837	88	2,146	744	0.346 692	0.653 308
46	73,345	848	0.011 562	0.988 438	89	1,402	555	0.395 863	0.604 137
47	72,497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71,627	896	0.012 509	0.987 491	91	462	246	0.532 466	0.467 534
49	70,731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69,804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68,842	1,011	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67,841	1,044	0.015 389	0.984 611	95	3	3	1.000 000	0.000 000

TABLE X. COMMUTATION COLUMNS, SINGLE PREMIUMS, AND ANNUITIES DUE, AMERICAN EXPERIENCE TABLE, 3½ PER CENT

Age x	D_x	N_x	C_x	M_x	$1 + a_x$	A_x
10	70891.9	1575 535.	513.02	17612.9	22.2245	0.24845
11	67981.5	1504 643.	493.69	17099.9	22.1331	0.25154
12	65189.0	1436 662.	475.08	16606.2	22.0384	0.25474
13	62509.4	1371 473.	457.16	16131.1	21.9403	0.25806
14	59938.4	1308 963.	439.91	15674.0	21.8385	0.26151
15	57471.6	1249 025.	423.88	15234.1	21.7329	0.26508
16	55104.2	1191 553.	407.87	14810.2	21.6236	0.26877
17	52832.9	1136 449.	392.47	14402.3	21.5102	0.27261
18	50653.9	1083 616.	378.15	14009.8	21.3926	0.27659
19	48562.8	1032 962.	364.36	13631.7	21.2707	0.28071
20	46556.2	984 400.	351.07	13267.3	21.1443	0.28497
21	44630.8	937 843.	338.73	12916.3	21.0134	0.28940
22	42782.8	893 213.	326.82	12577.5	20.8779	0.29399
23	41009.2	850 430.	315.33	12250.7	20.7375	0.29873
24	39307.1	809 421.	304.24	11935.4	20.5922	0.30365
25	37673.6	770 113.	293.55	11631.1	20.4417	0.30873
26	36106.1	732 440.	283.62	11337.6	20.2858	0.31401
27	34601.5	696 334.	274.03	11054.0	20.1244	0.31947
28	33157.4	661 732.	264.76	10779.9	19.9573	0.32512
29	31771.3	628 575.	256.16	10515.2	19.7843	0.33097
30	30440.8	596 804.	247.85	10259.0	19.6054	0.33702
31	29163.5	566 363.	239.797	10011.2	19.4202	0.34328
32	27937.5	537 199.	232.331	9771.38	19.2286	0.34976
33	26760.5	509 262.	225.406	9539.04	19.0304	0.35646
34	25630.1	482 501.	218.683	9313.64	18.8256	0.36339
35	24544.7	456 871.	212.157	9094.96	18.6138	0.37055
36	23502.5	432 326.	206.383	8882.80	18.3949	0.37795
37	22501.4	408 824.	200.757	8676.42	18.1688	0.38560
38	21539.7	386 323.	195.798	8475.66	17.9354	0.39349
39	20615.5	364 783.	190.945	8279.86	17.6946	0.40163
40	19727.4	344 167.	186.684	8088.92	17.4461	0.41003
41	18873.6	324 440.	182.493	7902.23	17.1901	0.41869
42	18052.9	305 566.	178.828	7719.74	16.9262	0.42762
43	17263.6	287 513.	175.421	7540.91	16.6543	0.43681
44	16504.4	270 250.	172.680	7365.49	16.3744	0.44628
45	15773.6	253 745.	170.127	7192.81	16.0867	0.45600
46	15070.0	237 972.	168.345	7022.68	15.7911	0.46600
47	14392.1	222 902.	166.872	6854.34	15.4878	0.47626
48	13738.5	208 510.	166.047	6687.47	15.1770	0.48677
49	13107.9	194 771.	165.983	6521.42	14.8591	0.49752
50	12498.6	181 663.	166.424	6355.44	14.5346	0.50849
51	11909.6	169 165.	167.316	6189.01	14.2041	0.51967
52	11339.5	157 252.	168.601	6021.70	13.8679	0.53104

TABLE X. COMMUTATION COLUMNS, SINGLE PREMIUMS, AND ANNUITIES DUE, AMERICAN EXPERIENCE TABLE, 3½ PER CENT—*Continued*

Age x	D_x	N_x	C_x	M_x	$1 + a_x$	A_x
53	10787.4	145916.	170.234	5853.10	13.5264	0.54258
54	10252.4	135128.	172.317	5682.86	13.1801	0.55430
55	9733.40	124876.	174.646	5510.54	12.8296	0.56615
56	9229.60	115142.	177.325	5335.90	12.4753	0.57813
57	8740.17	105912.8	180.168	5158.57	12.1179	0.59022
58	8264.44	97172.6	183.139	4978.40	11.7579	0.60239
59	7801.82	88908.2	186.340	4795.27	11.3958	0.61463
60	7351.65	81106.4	189.604	4608.93	11.0324	0.62692
61	6913.44	73754.7	192.909	4419.32	10.6683	0.63924
62	6486.75	66841.3	196.117	4226.41	10.3043	0.65155
63	6071.27	60354.5	199.109	4030.30	9.9410	0.66383
64	5666.85	54283.3	201.887	3831.19	9.5791	0.67607
65	5273.33	48616.4	204.457	3629.30	9.2193	0.68824
66	4890.55	43343.1	206.522	3424.84	8.8626	0.70030
67	4518.65	38452.5	208.022	3218.32	8.5097	0.71223
68	4157.82	33933.9	208.903	3010.30	8.1615	0.72401
69	3808.32	29776.1	208.858	2801.40	7.8187	0.73560
70	3470.67	25967.7	207.881	2592.54	7.4820	0.74698
71	3145.43	22497.1	205.639	2384.66	7.1523	0.75813
72	2833.42	19351.6	201.851	2179.02	6.8298	0.76904
73	2535.75	16518.2	196.436	1977.17	6.5141	0.77972
74	2253.57	13982.5	189.491	1780.73	6.2046	0.79018
75	1987.87	11728.9	181.253	1591.24	5.9002	0.80048
76	1739.39	9741.02	171.940	1409.99	5.6002	0.81062
77	1508.63	8001.63	161.889	1238.05	5.3039	0.82064
78	1295.73	6493.00	151.2646	1076.158	5.0111	0.83054
79	1100.647	5197.27	140.0891	924.894	4.7220	0.84032
80	923.338	4096.62	128.8801	784.805	4.4368	0.84997
81	763.234	3173.29	116.9588	655.924	4.1577	0.85940
82	620.465	2410.05	104.4881	538.966	3.8843	0.86865
83	494.995	1789.59	91.6152	434.478	3.6154	0.87774
84	386.641	1294.59	78.9565	342.862	3.3483	0.88677
85	294.610	907.95	67.0490	263.906	3.0819	0.89578
86	217.598	613.34	55.8566	196.857	2.8187	0.90468
87	154.383	395.74	45.1992	141.000	2.5634	0.91332
88	103.963	241.36	34.82426	95.8011	2.3216	0.92149
89	65.6231	137.398	25.09929	60.9768	2.0937	0.92920
90	38.3047	71.775	16.82244	35.8775	1.8738	0.93664
91	20.18692	33.4700	10.385393	19.05509	1.6580	0.94393
92	9.11888	13.2831	5.588150	8.66970	1.4567	0.95074
93	3.22236	4.16420	2.285484	3.08155	1.2923	0.95630
94	0.827611	0.94184	0.685393	0.79576	1.1380	0.96152
95	0.114232	0.114232	0.110369	0.110369	1.0000	0.96618

TABLE XI. VALUATION COLUMNS, AMERICAN EXPERIENCE TABLE, 3½ PER CENT

$$u_x = \frac{D_x}{D_{x+1}}, \quad k_x = \frac{C_x}{D_{x+1}}$$

Age x	u_x	k_x	Age x	u_x	k_x
10	1.042 811	0.007 546	53	1.052 185	0.016 604
11	1.042 838	0.007 573	54	1.053 323	0.017 704
12	1.042 866	0.007 600	55	1.054 585	0.018 922
13	1.042 894	0.007 627	56	1.055 999	0.020 289
14	1.042 922	0.007 654	57	1.057 563	0.021 800
15	1.042 962	0.007 692	58	1.059 296	0.023 474
16	1.042 990	0.007 720	59	1.061 234	0.025 347
17	1.043 019	0.007 748	60	1.063 385	0.027 425
18	1.043 059	0.007 787	61	1.065 780	0.029 739
19	1.043 100	0.007 826	62	1.068 433	0.032 303
20	1.043 141	0.007 866	63	1.071 365	0.035 136
21	1.043 195	0.007 917	64	1.074 625	0.038 285
22	1.043 248	0.007 969	65	1.078 270	0.041 807
23	1.043 303	0.008 022	66	1.082 304	0.045 704
24	1.043 358	0.008 076	67	1.086 782	0.050 031
25	1.043 415	0.008 130	68	1.091 774	0.054 855
26	1.043 484	0.008 197	69	1.097 284	0.060 178
27	1.043 554	0.008 264	70	1.103 403	0.066 090
28	1.043 625	0.008 333	71	1.110 117	0.072 576
29	1.043 710	0.008 415	72	1.117 388	0.079 602
30	1.043 796	0.008 498	73	1.125 218	0.087 167
31	1.043 884	0.008 583	74	1.133 660	0.095 323
32	1.043 986	0.008 682	75	1.142 852	0.104 204
33	1.044 102	0.008 795	76	1.152 960	0.113 971
34	1.044 221	0.008 910	77	1.164 314	0.124 941
35	1.044 343	0.009 027	78	1.177 243	0.137 433
36	1.044 493	0.009 172	79	1.192 031	0.151 720
37	1.044 647	0.009 320	80	1.209 771	0.168 861
38	1.044 830	0.009 498	81	1.230 099	0.188 502
39	1.045 018	0.009 679	82	1.253 477	0.211 089
40	1.045 238	0.009 891	83	1.280 245	0.236 952
41	1.045 463	0.010 109	84	1.312 384	0.268 004
42	1.045 721	0.010 359	85	1.353 917	0.308 133
43	1.046 001	0.010 629	86	1.409 469	0.361 806
44	1.046 331	0.010 947	87	1.484 979	0.434 762
45	1.046 684	0.011 289	88	1.584 244	0.530 671
46	1.047 106	0.011 697	89	1.713 188	0.655 254
47	1.047 571	0.012 146	90	1.897 500	0.833 333
48	1.048 111	0.012 668	91	2.213 750	1.138 889
49	1.048 745	0.013 280	92	2.829 873	1.734 177
50	1.049 463	0.013 974	93	3.893 571	2.761 905
51	1.050 272	0.014 755	94	7.245 000	6.000 000
52	1.051 177	0.015 629	95		

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